J. Differential Equations 251 (2011) 941–969

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Dispersive estimates for hyperbolic systems with time-dependent coefficients

Michael Ruzhansky [a,1], Jens Wirth [b,*,2]

[a] *Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK*

[b] *Institut für Analysis, Dynamik und Modellierung, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany*

ARTICLE INFO

Article history:
Received 14 May 2010
Revised 18 April 2011
Available online 13 May 2011

MSC:
35L05
35L15

Keywords:
Hyperbolic systems
Diagonalisation
Dispersive estimates
Oscillating time-dependent coefficients
van der Corput lemma

ABSTRACT

This paper is devoted to the study of time-dependent hyperbolic systems and the derivation of dispersive estimates for their solutions. It is based on a diagonalisation of the full symbol within adapted symbol classes in order to extract the essential information about representations of solutions. This is combined with a multi-dimensional van der Corput lemma to derive dispersive estimates.

1. Introduction

Our aim is to consider Cauchy problems for hyperbolic systems

$$\mathrm{D}_t U = A(t,\mathrm{D})U, \qquad U(0,\cdot) = U_0, \tag{1}$$

where $A(t,\mathrm{D})$ denotes a smoothly time-dependent matrix Fourier multiplier with first order symbol

$$A(t,\xi) \in C^\infty\big(\mathbb{R}_+ \times \mathbb{R}^n_\xi, \mathbb{C}^{m\times m}\big) \cap L^\infty\big(\mathbb{R}_+, S^1(\mathbb{R}^n)\big) \tag{2}$$

* Corresponding author.
E-mail addresses: m.ruzhansky@imperial.ac.uk (M. Ruzhansky), jens.wirth@iadm.uni-stuttgart.de (J. Wirth).

1 The first author was supported in part by the EPSRC grants EP/G007233/1 and EP/E062873/1.
2 The second author was supported by the EPSRC grant EP/E062873/1.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.04.022

subject to certain (natural) assumptions which are described later on in detail. As usual we denote $\mathrm{D}_t = -\mathrm{i}\partial_t$ and we assume that eigenvalues of A are real modulo 'lower order terms'.

Dispersive estimates for the wave equation go back to [2] and [16]. The decay of solutions to the scalar higher order equations with constant coefficients of dissipative type was considered in [13]. Consequently, a comprehensive analysis of the dispersive and Strichartz estimates of scalar hyperbolic equations of higher orders with constant coefficients, of general form, as well as for hyperbolic systems with constant coefficients, with applications to nonlinear equations was carried out in [14]. In analogy to this, the results of this paper have the corresponding (by now standard) consequences for Strichartz estimates, and for the global in time well-posedness of nonlinear equations, which we omit here, and refer to [14] for the corresponding constructions.

At the same time, equations of the second order with time-dependent coefficients have been intensively studied in the literature, see e.g. [10,11,8,19,21]. In [6], the asymptotic integration method was developed for scalar hyperbolic equations with time-dependent homogeneous symbols. The emphasis there was placed on minimising the time-differentiability assumptions on the symbol, required by applications to the Kirchhoff equations [5]. Indeed, the representation of solutions and dispersive estimates have been derived in [6] for equations of any order, by making assumptions on one time-derivative of the coefficients only, see Remark 21. However, the asymptotic integration produces an additional loss of regularity for high frequencies due to the fact that the amplitudes in the representation of solutions are symbols of type $(0,0)$. This is avoided in this paper since we construct the amplitudes as symbols of type $(1,0)$. The present paper generalises the treatment of [15] and provides proofs for the announced estimates. In [3] a special case of our results concerning differential hyperbolic systems with $\gamma = 2$ (i.e., with non-vanishing Gaussian curvature of the Fresnel surfaces) has been treated.

Our approach is based on diagonalising the (full) symbol of the operator, modulo controllable terms, in order to gain a representation of solutions in terms of Fourier integrals. The basic idea of this diagonalisation scheme comes from the treatment of degenerate hyperbolic problems suggested in the book of Yagdjian [22]; see also [20]. This representation is given in Theorem 20. Consequently, it is used to derive the dispersive estimates for solutions to the Cauchy problem (1) in Theorems 26 and 27, which are the main results of this paper. For this purpose, we derive more general estimates for oscillatory integrals in Propositions 24 and 25. The time decay rate of solutions to (1) is related to the geometric properties of the Fresnel surfaces of its characteristics in both convex and non-convex situations. In Section 4 we illustrate our assumptions for differential hyperbolic systems, for hyperbolic equations of the second order, and for general scalar hyperbolic equations of higher orders.

2. Prerequisites

The first step is based on a decomposition of the extended phase space $\mathbb{R}_+ \times \mathbb{R}^n$ into zones, a pseudo-differential zone containing bounded frequencies and a hyperbolic zone containing the large frequencies. It follows basically the treatment of [10,19], going back to the papers of Reissig and Yagdjian for the treatment of wave equations with time-dependent propagation speed ([8,7] and references cited therein, and in [10]) and the book of Yagdjian, [22], where the method of zones was introduced and applied to the investigation of the Cauchy problem for hyperbolic operators with multiple characteristics.

2.1. Hyperbolic symbol classes

We make use of the implicitly defined function t_ξ from

$$(1+t_\xi)|\xi| = N\big(\log(e+t_\xi)\big)^\nu \tag{2.1}$$

with suitable constants N and ν (usually required to belong to $[0,1]$) and define the zones

$$Z_{hyp}(N,\nu) = \big\{(t,\xi) \bigm| t \geqslant t_\xi\big\}, \qquad Z_{pd}(N) = \big\{(t,\xi) \bigm| 0 \leqslant t \leqslant t_\xi\big\}. \tag{2.2}$$

In $Z_{hyp}(N,\nu)$ we apply a diagonalisation procedure to the full symbol. The basic idea of this diagonalisation scheme comes from the treatment of degenerate hyperbolic problems and is closely related to the approach of [10]. If $\nu > 0$, the hyperbolic zone is subdivided further into two parts, the regular and the oscillating sub-zones

$$Z_{reg}(N,\nu) = Z_{hyp}(N,2\nu), \qquad Z_{osc}(N) = Z_{hyp}(N,\nu) \setminus Z_{hyp}(N,2\nu). \tag{2.3}$$

We denote the new boundary as $\tilde{t}_\xi$. It satisfies $(1+\tilde{t}_\xi)|\xi| = N(\log(e+\tilde{t}_\xi))^{2\nu}$. The parameter ν is used to control the allowed amount of oscillations in the time-dependence of coefficients and amplitude functions. Based on the terminology from [22, Section 3.11.1] and of [7,9], one speaks of *very slow oscillations* if $\nu = 0$, *slow oscillations* if $\nu \in (0,1)$, and *fast oscillations* if $\nu = 1$. The following definition of symbol classes and also the basic assumptions introduced later on will depend on the parameter ν.

Definition 1. The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the symbol class $\mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1,m_2\}$ if it satisfies the symbolic estimates

$$\left|\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)\right| \leqslant C_{k,\alpha} |\xi|_t^{m_1-|\alpha|} \left(\frac{1}{1+t}\big(\log(e+t)\big)^\nu\right)^{m_2+k} \tag{2.4}$$

for all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant \ell_1$ and all natural numbers $k \leqslant \ell_2$, where

$$|\xi|_t = \max\left(|\xi|, \frac{N}{1+t}\big(\log(e+t)\big)^\nu\right). \tag{2.5}$$

If the symbolic estimates hold for all derivatives we write $\mathcal{S}_{N,\nu}\{m_1,m_2\}$ short for $\mathcal{S}^{\infty,\infty}_{N,\nu}\{m_1,m_2\}$. Furthermore, the definition extends immediately to matrix-valued Fourier multipliers. The rules of the corresponding symbolic calculus are simple consequences of Definition 1 together with (2.1)–(2.3) and are collected in the following proposition.

Proposition 2.

(1) *$\mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1,m_2\}$ is a vector space.*
(2) *$\mathcal{S}^{\ell_1',\ell_2'}_{N,\nu}\{m_1-k,m_2+\ell\} \hookrightarrow \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1,m_2\}$ for all $\ell \geqslant k \geqslant 0$, $\ell_1' \geqslant \ell_1$ and $\ell_2' \geqslant \ell_2$.*
(3) *$\mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1,m_2\} \cdot \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1',m_2'\} \hookrightarrow \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1+m_1',m_2+m_2'\}$.*
(4) *$\mathrm{D}_t^k \mathrm{D}_\xi^\alpha \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{m_1,m_2\} \hookrightarrow \mathcal{S}^{\ell_1-|\alpha|,\ell_2-k}_{N,\nu}\{m_1-|\alpha|,m_2+k\}$.*
(5) *$\mathcal{S}^{0,0}_{N,\nu}\{-1,2\} \hookrightarrow L^\infty_\xi L^1_t(Z_{reg}(N,\nu))$.*

Proof. The statements are simple consequences of the symbolic estimates. Since it is vitally important for us later on, we show the last one. It follows that

$$\begin{aligned}
\int_t^\infty \left|a(\tau,\xi)\right| \mathrm{d}\tau &\lesssim \int_t^\infty \frac{(\log(e+\tau))^{2\nu}}{|\xi|(e+\tau)^2}\,\mathrm{d}\tau \\
&= \left[-\frac{(\log(e+\tau))^{2\nu}}{|\xi|(e+\tau)}\right]_t^\infty + 2\nu \int_t^\infty \frac{(\log(e+\tau))^{2\nu-1}}{|\xi|(e+\tau)^2}\,\mathrm{d}\tau \lesssim \frac{(\log(e+t))^{2\nu}}{|\xi|(1+t)} \lesssim 1
\end{aligned}$$

for any $a \in \mathcal{S}^{0,0}_{N,\nu}\{-1,2\}$ uniform in $(t,\xi) \in Z_{reg}(N,\nu)$. □

Of particular importance are the embedding relations (2). They constitute a symbolic hierarchy, which is used in the diagonalisation scheme, cf. Section 3.2.2. We define the residual symbol classes

$$\mathcal{H}_{N,\nu}\{m\} = \bigcap_{k\in\mathbb{Z}} \mathcal{S}_{N,\nu}\{m-k,k\} \tag{2.6}$$

as intersections along the embeddings $\mathcal{S}_{N,\nu}\{m_1-k, m_2+k\} \hookrightarrow \mathcal{S}_{N,\nu}\{m_1,m_2\}$ for $k \geqslant 0$.

For later use we define the cut-off function $\chi_{pd}(t,\xi) = \chi(|\xi|(1+t)(\log(e+t))^{-\nu}/N)$ for $\chi \in C_0^\infty(\mathbb{R})$, $\chi(s) = 1/2$ for $s \leqslant 1$ and $\chi(s) = 0$ for $s > 1$ and similarly $\chi_{hyp}(t,\xi) = 1 - \chi_{pd}(t,\xi)$.

2.2. Basic assumptions

We collect our assumptions on the symbol $A(t,\xi)$. Throughout this paper we require

(A1)$_{\ell_1,\ell_2}$ *Hyperbolic operator of first order with bounded coefficients.* We assume that the matrix operator $A(t,\mathrm{D})$ has a symbol satisfying

$$A(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{1,0\} \tag{2.7}$$

for some $N > 0$ and $\nu \in [0,1]$, which has all its eigenvalues in a strip $\operatorname{spec} A(t,\xi) \subseteq S_c = \{\zeta \in \mathbb{C}\colon\ |\operatorname{Im}\zeta| \leqslant c\}$ for all $(t,\xi) \in Z_{hyp}(N,\nu)$.

We say the system is *symmetric hyperbolic* if we assume in addition

$$\operatorname{Im} A(t,\xi) = \frac{1}{2\mathrm{i}}\big(A(t,\xi) - A^*(t,\xi)\big) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{0,1\}. \tag{2.8}$$

We write **(A1$^+$)** for this stronger assumption.

(A2) *Uniform strict hyperbolicity up to $t = \infty$.* We assume that there exists a homogeneous of order one in ξ matrix $A_1(t,\xi)$ with $A(t,\xi) - A_1(t,\xi)\chi_{hyp}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{0,1\}$, which has real and distinct eigenvalues

$$\lambda_1(t,\xi),\ldots,\lambda_m(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{1,0\},$$

the so-called *characteristic roots* of the symbol $A_1(t,\xi)$. In ascending order we denote them as $\lambda_1(t,\xi),\ldots,\lambda_m(t,\xi)$. Furthermore, we assume that

$$\liminf_{t\to\infty} \min_{\omega\in\mathbb{S}^{n-1}} \big|\lambda_i(t,\omega) - \lambda_j(t,\omega)\big| > 0 \tag{2.9}$$

for all $i \neq j$.

As a consequence of the first two assumptions we get

Proposition 3. *Assume* (A1)$_{\ell_1,\ell_2}$ *and* (A2)*. For all $j = 1,\ldots,m$ the characteristic roots satisfy $\lambda_j(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{1,0\}$ and for all $i \neq j$ their difference satisfies $(\lambda_i(t,\xi) - \lambda_j(t,\xi))^{-1} \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{-1,0\}$. Furthermore, the eigenprojection $P_j(t,\xi)$ corresponding to $\lambda_j(t,\xi)$ satisfies $P_j(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2}\{0,0\}$.*

Proof *(Sketch).* The properties of the characteristic roots follow from the spectral estimate $|\lambda_j(t,\omega)| \leqslant \|A_1(t,\omega)\|$ together with the obvious symbol properties of the coefficients of the characteristic polynomial and the uniform strict hyperbolicity. The eigenprojections can be expressed in terms of the characteristic roots

$$P_j(t,\xi) = \prod_{i\neq j} \frac{A_1(t,\xi) - \lambda_i(t,\xi)\mathrm{I}}{\lambda_j(t,\xi) - \lambda_i(t,\xi)} \tag{2.10}$$

and again the symbolic calculus yields the desired result. □

For $a_1, \dots, a_m \in \mathbb{C}$ we define $\operatorname{diag}(a_1, \dots, a_m)$ to be the diagonal matrix with entries $a_1, \dots, a_m$ along the diagonal. For a matrix M, we define $\operatorname{diag} M$ as the diagonal matrix with $M_{11}, \dots, M_{mm}$ at the diagonal.

Proposition 4. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$ *and* (A2). *There exists a uniformly bounded and invertible matrix* $M(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$ *with* $M^{-1}(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$ *which diagonalises the symbol* $A_1(t,\xi)$,

$$A_1(t,\xi)M(t,\omega) = M(t,\xi)\mathcal{D}(t,\xi), \quad \mathcal{D}(t,\xi) = \operatorname{diag}\bigl(\lambda_1(t,\xi), \dots, \lambda_m(t,\xi)\bigr). \tag{2.11}$$

Proof. By Proposition 3 we see that the symmetriser of $A_1(t,\xi)$ given as $H(t,\xi) = \sum_j P_j^*(t,\xi)P_j(t,\xi)$ satisfies $H(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$. Our aim is to express $M^{-1}(t,\xi)$ explicitly in terms of the eigenprojections $P_j(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$. Let therefore $v_j(t,\xi)$ be (a smoothly chosen) unit vector from the one-dimensional j-th eigenspace $\operatorname{ran} P_j(t,\xi)$ and $M^{-1}(t,\xi)u$ be the vector with the entries given by the scalar products $(v_j, P_j(t,\xi)u) = (v_j, H(t,\xi)u)$ for any vector $u \in \mathbb{C}^n$.

Note that $v_j(t,\xi)$ is unique up to a sign and locally in t and ξ expressible as $v_j(t,\xi) = P_j(t,\xi)e/\|P_j(t,\xi)e\|$ for some fixed unit vector e chosen away from the orthogonal complement to the eigenspace. Differentiating this yields the desired bounds $v_j(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$ and therefore $M^{-1}(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$. The inverse matrix $M(t,\xi)$ has the vectors $v_j(t,\xi)$ as its columns and therefore $M(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}_{N,\nu}\{0,0\}$ as claimed. $\square$

Remark 5. Under Assumption $(\mathrm{A1}^+)$ the matrix $M(t,\xi)$ can be chosen unitary, which simplifies some of the considerations later on.

(A3) *Generalised energy conservation property.* We have to make a technical assumption which guarantees us that the lower order terms are not too strong. This can be written as

$$\sup_{(s,\xi),(t,\xi)\in Z_{hyp}(N,\nu)} \left\| \int_s^t \operatorname{Im} F_0(\theta,\xi)\,\mathrm{d}\theta \right\| < \infty \tag{2.12}$$

for the matrix

$$F_0(t,\xi) = \operatorname{diag}\bigl(M^{-1}(t,\xi)\bigl(A(t,\xi) - A_1(t,\xi)\bigr)M(t,\xi) - M^{-1}(t,\xi)\bigl(\mathrm{D}_t M(t,\xi)\bigr)\bigr). \tag{2.13}$$

Remark 6. This statement does not follow from symbol estimates as we only have $F_0 \in \mathcal{S}^{\ell_1,\ell_2-1}_{N,\nu}\{0,1\}$. We will later use it in Theorem 15 relating it to uniform bounds on the micro-energy of solutions within $Z_{reg}(N,\nu)$ explaining the nature of (A3). Assumption (A3) is independent of the choice of the diagonaliser $M(t,\xi)$, provided it satisfies the properties of Proposition 4.

If we assume $(\mathrm{A1}^+)$, then the diagonaliser $M(t,\xi)$ can be chosen unitary. This implies

Proposition 7. *Assume* $(\mathrm{A1}^+)$. *Then the matrix* $M^*(\mathrm{D}_t M)$ *is self-adjoint and therefore*

$$\operatorname{Im} \operatorname{diag} M^*(t,\xi)\mathrm{D}_t M(t,\xi) = 0, \tag{2.14}$$

in particular, $\operatorname{Im} F_0(t,\xi) = \operatorname{Im} \operatorname{diag}(M^*(t,\xi)(A(t,\xi) - A_1(t,\xi))M(t,\xi))$.

Proof. Obviously $0 = \mathrm{D}_t \mathrm{I} = (\mathrm{D}_t M^*)M + M^*(\mathrm{D}_t M)$ and therefore

$$\bigl(M^*\mathrm{D}_t M\bigr)^* = (\mathrm{D}_t M)^* M = -\bigl(\mathrm{D}_t M^*\bigr)M = M^*(\mathrm{D}_t M).$$

But this implies that the diagonal entries of $(\mathrm{D}_t M^*)M$ are real. $\square$

Assumption (A3) can be replaced by the weaker assumption that there exists a real scalar function $\delta(t,\xi) \in \mathcal{S}\{0,1\}$ such that $\operatorname{Im} F_0(t,\xi) - \delta(t,\xi)\mathrm{I}$ satisfies the equivalent of (2.12), i.e.

$$\sup_{(s,\xi),(t,\xi)\in Z_{reg}(N,\nu)} \left\| \int_s^t \big(\operatorname{Im} F_0(\theta,\xi) - \delta(\theta,\xi)\mathrm{I}\big)\,\mathrm{d}\theta \right\| < \infty. \tag{2.15}$$

We refer to this weaker assumption as **(A3$^-$)**.

(A4) *Weak dissipativity.* There exist a constant c and a function $\gamma(t)$ such that

$$\operatorname{Im} A(t,\xi) + c|\xi|\mathrm{I} + \frac{1}{2}\gamma(t)\mathrm{I} \geqslant 0 \tag{2.16}$$

in the sense of matrices and within $Z_{pd}(N,\nu)$ for sufficiently large N, where

$$\int_0^t \gamma(s)\,\mathrm{d}s \leqslant C\big(\log(e+t)\big)^{\nu} \tag{2.17}$$

for some constant C and all $t \geqslant 0$.

If we assume (A3$^-$), we also weaken (A4) to **(A4$^-$)**

$$\operatorname{Im} A(t,\xi) + c|\xi|\mathrm{I} + \frac{1}{2}\gamma(t)\mathrm{I} + \frac{1}{2}\delta(t,\xi)\mathrm{I} \geqslant 0 \tag{2.18}$$

with the same $\delta(t,\xi)$ as in (A3$^-$).

Remark 8. This assumption is mainly for convenience and to simplify the main estimate in the pseudo-differential zone given in Lemma 9. If one wants to obtain sharp estimates in specific situations a more detailed analysis of the behaviour of solutions for small frequencies has to be carried out. We refer to [19] for an example, where such a more detailed consideration was essential.

Another suitable replacement for (A4) could be to assume that the full symbol $A(t,\xi)$ has all its eigenvalues in the (closed) complex upper half plane and that it is uniformly symmetrisable on $Z_{pd}(N',\nu)$.

Later on we will discuss the construction of representations of solutions and resulting estimates under Assumptions (A1) to (A4). Under the weaker Assumptions (A3$^-$) and (A4$^-$) the uniform behaviour of the energy within $Z_{reg}(N,\nu)$ is replaced by an equivalence to $\exp(\int_0^t \delta(\tau,\xi)\,\mathrm{d}\tau)$ and can be obtained by minor modifications in the arguments.

3. Representation of solutions

In order to solve (1) we apply a partial Fourier transform with respect to the spatial variables. Assuming $U_0 \in \mathcal{S}(\mathbb{R}^n)$, this gives a system of ordinary differential equations

$$\mathrm{D}_t\widehat{U} = A(t,\xi)\widehat{U}, \qquad \widehat{U}(0,\cdot) = \widehat{U}_0 \tag{3.1}$$

parametrised by the frequency parameter $\xi \in \mathbb{R}^n$ with initial data $\widehat{U}_0 \in \mathcal{S}(\mathbb{R}^n)$. We construct its fundamental solution $\mathcal{E}(t,s,\xi)$, i.e. the solution to the matrix-valued problem

$$\mathrm{D}_t\mathcal{E}(t,s,\xi)=A(t,\xi)\mathcal{E}(t,s,\xi),\quad \mathcal{E}(s,s,\xi)=\mathrm{I}\in\mathbb{C}^{m\times m}. \tag{3.2}$$

Then, the inverse partial Fourier transform gives $U(t,x)=\mathcal{E}(t,0,\mathrm{D})U_0(x)$ for all t and x, while our construction gives a representation of $\mathcal{E}(t,0,\mathrm{D})$ as a matrix of Fourier integral operators.

3.1. Treatment in the pseudo-differential zone

We use the dissipativity Assumption (A4) and restrict considerations to the zone $Z_{pd}(N,\nu)$.

Lemma 9. *Assume* (A4)*. Then the fundamental solution to* (3.2) *satisfies*

$$\big\|\mathcal{E}(t,0,\xi)\big\|\leqslant C\exp\big(C'\big(\log(e+t_\xi)\big)^\nu\big) \tag{3.3}$$

for some constants C and C' and uniform in $(t,\xi)\in Z_{pd}(N,\nu)$.

Proof. Fix $\xi\in\mathbb{R}^n$ and denote by $V(t)$ the solution to $\mathrm{D}_tV(t)=A(t,\xi)V(t)$ with data $V(0)=V_0$. Then using the Euclidean inner product $(\cdot,\cdot)$ on $\mathbb{C}^m$ we obtain from (A4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|V(t)\big\|^2=-2\big(\operatorname{Im}A(t,\xi)V(t),V(t)\big)\leqslant 2c|\xi|\big\|V(t)\big\|^2+\gamma(t)\big\|V(t)\big\|^2 \tag{3.4}$$

for all $t\leqslant t_\xi$. Hence, by applying the Gronwall inequality the estimate

$$\big\|V(t)\big\|^2\lesssim\|V_0\|^2\exp\left(2ct|\xi|+\int_0^t\gamma(s)\,\mathrm{d}s\right)\lesssim\exp\big(C'\big(\log(e+t_\xi)\big)^\nu\big) \tag{3.5}$$

follows. This proves the statement. $\square$

Later on we need estimates for derivatives of $\mathcal{E}(t_\xi,0,\xi)$ with respect to ξ as $|\xi|\to 0$. They essentially follow from the symbol estimates satisfied by $A(t,\xi)$ due to $(\mathrm{A1})_{\ell_1,\ell_2}$.

Lemma 10. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$ *and* (A4)*. For all multi-indices $|\alpha|\leqslant\min(\ell_1,\ell_2)$ the estimate*

$$\big\|\mathrm{D}_\xi^\alpha\mathcal{E}(t_\xi,0,\xi)\big\|\leqslant C_\alpha|\xi|^{-|\alpha|}\big(\log(e+t_\xi)\big)^{\nu|\alpha|}\exp\big(C'\big(\log(e+t_\xi)\big)^\nu\big) \tag{3.6}$$

is valid uniformly on $|\xi|\leqslant N$.

Proof. To prove this fact we use Duhamel's formula for ξ-derivatives of (3.2). Let first $|\alpha|=1$. Then $\mathrm{D}_t\mathrm{D}_\xi^\alpha\mathcal{E}=(\mathrm{D}_\xi^\alpha A)\mathcal{E}+A(\mathrm{D}_\xi^\alpha\mathcal{E})$ and thus using $\mathrm{D}_\xi^\alpha\mathcal{E}(0,0,\xi)=0$ we obtain the representation

$$\mathrm{D}_\xi^\alpha\mathcal{E}(t,0,\xi)=\int_0^t\mathcal{E}(t,\tau,\xi)\big(\mathrm{D}_\xi^\alpha A(\tau,\xi)\big)\mathcal{E}(\tau,0,\xi)\,\mathrm{d}\tau. \tag{3.7}$$

Since $\|\mathrm{D}_\xi^\alpha A(\tau,\xi)\|\lesssim 1$, this implies the estimate

$$\begin{aligned}\big\|\mathrm{D}_\xi^\alpha\mathcal{E}(t,0,\xi)\big\|&\lesssim t\exp\big(C\big(\log(e+t)\big)^\nu\big)\\&\lesssim|\xi|^{-1}\big(\log(e+t_\xi)\big)^\nu\exp\big(C\big(\log(e+t_\xi)\big)^\nu\big)\end{aligned} \tag{3.8}$$

uniformly on $Z_{pd}(N,\nu)$.

For $|\alpha| = \ell > 1$ we use Leibniz formula to represent $\mathrm{D}_\xi^\alpha \mathcal{E}(t,0,\xi)$ by a corresponding Duhamel integral using a sum of terms $(\mathrm{D}_\xi^{\alpha_1} A(t,\xi))(\mathrm{D}_\xi^{\alpha_2}\mathcal{E}(t,0,\xi))$ for $|\alpha_1| + |\alpha_2| \leqslant \ell$, $\alpha_2 < \alpha$, and apply induction over ℓ to obtain

$$\big\|\mathrm{D}_\xi^\alpha \mathcal{E}(t,0,\xi)\big\| \lesssim |\xi|^{-|\alpha|}\big(\log(e+t_\xi)\big)^{\nu|\alpha|} \exp\big(C'\big(\log(e+t_\xi)\big)^\nu\big), \quad |\xi| \leqslant N. \tag{3.9}$$

Derivatives of $\mathcal{E}$ with respect to t are estimated directly by the differential equation and $\|A(t,\xi)\| \lesssim |\xi|$. Combined with

$$\big|\mathrm{D}_\xi^\alpha t_\xi\big| \lesssim t_\xi |\xi|^{-\alpha} \lesssim |\xi|^{-1-|\alpha|}\big(\log(e+t_\xi)\big)^\nu \tag{3.10}$$

this completes the proof. □

Remark 11. If one is only interested in energy estimates, the statement of Lemma 9 is sufficient to get the corresponding micro-energy estimate in the pseudo-differential zone.

Remark 12. Any replacement of (A4), which is sufficient to deduce Lemma 9, is also enough to prove Lemma 10. This is interesting in particular for the treatment for second order scalar problems, where other ways to prove the estimate of Lemma 9 are available and important to obtain sharp result. See, e.g., the treatment in [19] or [4] where reformulations as integral equations were used.

3.2. Treatment in the hyperbolic zone

The following considerations are only applied within the hyperbolic zone. We can write formulae globally for all t and ξ if we make use of the cut-off function $\chi_{hyp}(t,\xi)$. We omit this to keep notation as simple as possible.

3.2.1. Treatment of the main part

We apply the diagonaliser $M(t,\xi)$ of $A_1(t,\xi)$. For this we set $U^{(0)} = M^{-1}(t,\xi)\widehat{U}$, so that

$$\begin{aligned}
\mathrm{D}_t U^{(0)} &= \big(\mathrm{D}_t M^{-1}(t,\xi)\big)\widehat{U} + M^{-1}(t,\xi)\mathrm{D}_t\widehat{U} \\
&= \big(\big(\mathrm{D}_t M^{-1}\big)M + M^{-1}(t,\xi)A(t,\xi)M(t,\xi)\big)U^{(0)} \\
&= \big(\mathcal{D}(t,\xi) + R_0(t,\xi)\big)U^{(0)}
\end{aligned} \tag{3.11}$$

is valid. Now the main term $\mathcal{D}(t,\xi) = \operatorname{diag}(\lambda_1(t,\xi),\dots,\lambda_m(t,\xi))$ is diagonal and the remainder is given by

$$R_0(t,\xi) = M^{-1}(t,\xi)\big(A(t,\xi) - A_1(t,\xi)\big)M(t,\xi) + \big(\mathrm{D}_t M^{-1}(t,\xi)\big)M(t,\xi). \tag{3.12}$$

The rules of the symbolic calculus imply $R_0(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-1}\{0,1\}$, while our assumptions on the characteristic roots give $\mathcal{D}(t,\xi) \in \mathcal{S}_{N,\nu}^{\infty,\ell_2}\{1,0\}$.

3.2.2. The diagonalisation scheme

The initial diagonalisation step was done in the previous section. We transformed the system by a matrix $M \in \mathcal{S}_{N,\nu}^{\infty,\ell_2}\{0,0\}$ to a diagonal form modulo $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-1}\{0,1\}$. In the k-th step we want to diagonalise it modulo the class $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$ for sufficiently large N. (The choice of N depends on k in order to guarantee the invertibility of the transformation matrix $N_k \in \mathcal{S}_{N,\nu}\{0,0\}$ below.)

Recursively, we construct matrices $N_k(t,\xi)$ to diagonalise the system $\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)$ modulo $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$. We use the notation

$$N_k(t,\xi) = \sum_{j=0}^{k} N^{(j)}(t,\xi), \qquad F_k(t,\xi) = \sum_{j=0}^{k} F^{(j)}(t,\xi), \tag{3.13}$$

where initially $N^{(0)}(t,\xi) = \mathrm{I}$, $B^{(0)}(t,\xi) = R_0(t,\xi)$ and $F^{(0)}(t,\xi) = \operatorname{diag} B^{(0)}(t,\xi)$. The construction goes along the following recursive scheme.

Step *k*, $k \geqslant 1$. We set

$$F^{(k-1)}(t,\xi) = \operatorname{diag} B^{(k-1)}(t,\xi), \tag{3.14}$$

$$\big(N^{(k)}(t,\xi)\big)_{i,j} = \begin{cases} \frac{-B^{(k-1)}(t,\xi)_{ij}}{\lambda_i(t,\xi)-\lambda_j(t,\xi)}, & i \neq j, \\ 0, & i = j, \end{cases} \tag{3.15}$$

$$\begin{aligned} B^{(k)}(t,\xi) = {} & \big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)\big) N_k(t,\xi) \\ & - N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - F_{k-1}(t,\xi)\big). \end{aligned} \tag{3.16}$$

We prove by induction over k the estimates $N^{(k)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{-k,k\}$ and $B^{(k)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$. For $k=1$ we know

$$\begin{aligned} F^{(0)}(t,\xi) &\in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-1}\{0,1\}, \\ N^{(1)}(t,\xi) &\in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-1}\{-1,1\}, \\ B^{(1)}(t,\xi) &\in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-2}\{-1,2\}, \end{aligned} \tag{3.17}$$

the last one follows from the representation

$$B^{(1)}(t,\xi) = \mathrm{D}_t N^{(1)}(t,\xi) - \big(R_0(t,\xi) - F^{(0)}(t,\xi)\big) N^{(1)}(t,\xi).$$

For $k \geqslant 1$, assume we know $B^{(k-1)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{-k+1,k\}$. Then by definition of $N^{(k)}$ we have from $(\lambda_i(t,\xi) - \lambda_j(t,\xi))^{-1} \in \mathcal{S}_{N,\nu}^{\infty,\ell_2}\{-1,0\}$ that $N^{(k)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{-k,k\}$ and obviously $F^{(k-1)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{-k+1,k\}$. Moreover, by algebraic manipulations we get

$$\begin{aligned} B^{(k)} &= (\mathrm{D}_t - \mathcal{D} - R_0)\left(\sum_{\nu=0}^{k} N^{(\nu)}\right) - \left(\sum_{\nu=0}^{k} N^{(\nu)}\right)\left(\mathrm{D}_t - \mathcal{D} - \sum_{\nu=0}^{k-1} F^{(\nu)}\right) \\ &= B^{(k-1)} + \big[N^{(k)},\mathcal{D}\big] - F^{(k-1)} + \mathrm{D}_t N^{(k)} + R_0 N^{(k)} \\ &\quad + N^{(k)} \sum_{\nu=0}^{k-1} F^{(\nu)} - \left(\sum_{\nu=1}^{k} N^{(\nu)}\right) F^{(k)}. \end{aligned}$$

The definition of $N^{(k)}(t,\xi)$ was chosen in such a way that we have

$$B^{(k-1)}(t,\xi) + \big[N^{(k)}(t,\xi),\mathcal{D}(t,\xi)\big] - F^{(k-1)}(t,\xi) = 0. \tag{3.18}$$

The sum of the remaining terms belongs to the symbol class $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$. Hence we have proved $B^{(k)}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$. Furthermore, the definition of $B^{(k)}(t,\xi)$ implies the operator identity

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)\big) N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - F_{k-1}(t,\xi)\big) \tag{3.19}$$

modulo $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k, k+1\}$. Thus we have constructed the desired diagonaliser if we can show that the matrix $N_k(t,\xi)$ is invertible on $Z_{hyp}(N,\nu)$ with uniformly bounded inverse. But this follows from $N_k - \mathrm{I} \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{-1, 1\}$ by the choice of a sufficiently large zone constant N. Indeed, we have

$$\big\| N_k(t,\xi) - \mathrm{I} \big\| \leqslant C \frac{(\log(e+t))^\nu}{|\xi|(1+t)} \leqslant \frac{C'}{N} \to 0 \quad \text{as } N \to \infty. \tag{3.20}$$

Thus with the notation $R_k(t,\xi) = -N_k^{-1}(t,\xi) B^{(k)}(t,\xi)$ we have proven the following lemma.

Lemma 13. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$ *and* (A2). *For each* $1 \leqslant k < \ell_2$ *there exist a zone constant* N *and matrix-valued symbols*

- $N_k(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{0,0\}$, *which is invertible for all* $(t,\xi) \in Z_{hyp}(N,\nu)$ *and its inverse satisfies* $N_k^{-1}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{0,0\}$;
- $F_{k-1}(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{0,1\}$, *which is diagonal*;
- $R_k(t,\xi) \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k-1}\{-k,k+1\}$;

such that the (*operator*) *identity*

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)\big) N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big) \tag{3.21}$$

holds true for all $(t,\xi) \in Z_{hyp}(N,\nu)$.

3.2.3. Remarks on perfect diagonalisation

Under Assumption $(\mathrm{A1})_{\infty,\infty}$ Lemma 13 can be understood as a perfect diagonalisation of the original system. If we define $F(t,\xi)$ as an asymptotic sum of the $F^{(k)}(t,\xi)$,

$$F(t,\xi) \sim \sum_{k=0}^{\infty} F^{(k)}(t,\xi), \tag{3.22}$$

this means we require $F(t,\xi) - F_k(t,\xi) \in \mathcal{S}_{N,\nu}\{-k-1, k+2\}$ for all $k \in \mathbb{N}$, and similarly

$$N(t,\xi) \sim \sum_{k=0}^{\infty} N^{(k)}(t,\xi), \tag{3.23}$$

which can be chosen to be invertible, Eq. (3.19) implies

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R(t,\xi)\big) N(t,\xi) - N(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - F(t,\xi)\big) \in \mathcal{H}_{N,\nu}\{1\}. \tag{3.24}$$

In this sense the symbol class $\mathcal{H}_{N,\nu}\{1\}$ is understood as the residual symbol class within the scheme of diagonalisation.

3.2.4. Solving the diagonalised system in the regular sub-zone

Our strategy is as follows. In the first step we solve the diagonal system $\mathrm{D}_t - \mathcal{D} - F_{k-1}$. Its fundamental solution $\widetilde{\mathcal{E}}_k(t,s,\xi)$ is given by

$$\widetilde{\mathcal{E}}_k(t,s,\xi) = \exp\left(\mathrm{i}\int_s^t \big(\mathcal{D}(\tau,\xi) + F_{k-1}(\tau,\xi)\big)\,\mathrm{d}\tau\right)$$

$$= \exp\left(\mathrm{i}\int_s^t \mathcal{D}(\tau,\xi)\,\mathrm{d}\tau\right) \exp\left(\mathrm{i}\int_s^t F_{k-1}(\tau,\xi)\,\mathrm{d}\tau\right). \tag{3.25}$$

The second factor is uniformly bounded with uniformly bounded inverse on $Z_{reg}(N,\nu)$, because $F_{k-1} - F^{(0)} \in \mathcal{S}^{0,0}_{N,\nu}\{-1,2\} \subseteq L^\infty_\xi L^1_t(Z_{reg}(N,\nu))$ by Proposition 2(5), and $\operatorname{Im} F^{(0)}$ satisfies (A3). The first factor is a unitary matrix. Thus, we obtain the uniform bound

$$\big\| \widetilde{\mathcal{E}}_k(t,s,\xi)\big\| \leqslant C, \quad (t,\xi),(s,\xi) \in Z_{reg}(N,\nu), \tag{3.26}$$

regardless of the order of s and t.

In the second step we make the *ansatz*

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi) \tag{3.27}$$

for the fundamental solution $\mathcal{E}_k(t,s,\xi)$ to $\mathrm{D}_t - \mathcal{D} - F_k - R_k$. Then the matrix $\mathcal{Q}_k(t,s,\xi)$ satisfies

$$\mathrm{D}_t\mathcal{Q}_k(t,s,\xi) = \mathcal{R}_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \quad \mathcal{Q}_k(s,s,\xi) = \mathrm{I} \in \mathbb{C}^{m\times m}, \tag{3.28}$$

with coefficient matrix

$$\mathcal{R}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(s,t,\xi)R_k(t,\xi)\widetilde{\mathcal{E}}_k(t,s,\xi). \tag{3.29}$$

$R_k \in \mathcal{S}_{N,\nu}\{-k,k+1\}$ with $k \geqslant 1$ implies uniform integrability of $\mathcal{R}_k$ over the regular sub-zone. Therefore, the representation of $\mathcal{Q}_k(t,s,\xi)$ as the Peano–Baker series

$$\mathcal{Q}_k(t,s,\xi) = \mathrm{I} + \sum_{k=1}^\infty \mathrm{i}^k \int_s^t \mathcal{R}_k(t_1,s,\xi) \int_s^{t_1} \mathcal{R}_k(t_2,s,\xi) \cdots \int_s^{t_{k-1}} \mathcal{R}_k(t_k,s,\xi)\,\mathrm{d}t_k\cdots\mathrm{d}t_1 \tag{3.30}$$

implies boundedness of $\mathcal{Q}_k(t,s,\xi)$ uniformly in $(t,\xi),(s,\xi) \in Z_{reg}(N,\nu)$. We obtain even a little bit more:

Lemma 14. *Assume* $(\mathrm{A1})_{0,\ell_2}$, (A2) *and* (A3). *Then the matrices* $\mathcal{Q}_k(t,s,\xi)$, $k < \ell_2$, *defined as solutions to* (3.28) *are uniformly bounded on* $Z_{hyp}(N,\nu)$, *invertible and tend locally uniformly in* $\xi \neq 0$ *to* $\mathcal{Q}_k(\infty,s,\xi) \in L^\infty(Z_{hyp}(N,\nu))$ *as* $t \to \infty$.

Proof. The convergence follows directly by applying the Cauchy criterion to the representation (3.30). Invertibility is a consequence of the Liouville theorem, which gives

$$\det \mathcal{Q}_k(t,s,\xi) = \exp\left(\mathrm{i}\int_s^t \operatorname{trace}\mathcal{R}_k(\theta,s,\xi)\,\mathrm{d}\theta\right) = \exp\left(\mathrm{i}\int_s^t \operatorname{trace} R_k(\theta,\xi)\,\mathrm{d}\theta\right)$$
$$\geqslant \exp\left(-m\int_s^t \left\|R_k(\theta,\xi)\right\|\,\mathrm{d}\theta\right) \geqslant C > 0, \tag{3.31}$$

uniform in $(t,\xi), (s,\xi) \in Z_{hyp}(N,\nu)$. $\square$

The following theorem explains the true nature of Assumption (A3) provided that $\nu = 0$.

Theorem 15. *Assume* $(\mathrm{A1})_{0,\infty}$ *with* $\nu = 0$ *and* (A2)*. Then* (A3) *is equivalent to the existence of a constant* $C > 0$ *such that*

$$C^{-1}\|V\| \leqslant \left\|\mathcal{E}(t,s,\xi)V\right\| \leqslant C\|V\| \tag{3.32}$$

for all vectors $V \in \mathbb{C}^m$ *and* $(t,\xi), (s,\xi) \in Z_{hyp}(N,0)$*.*

Proof. Lemma 14 in combination with (3.26) gives the uniform bound under (A3). Without (A3) the estimate of (3.26) has to be replaced by a polynomial bound

$$\left\|\widetilde{\mathcal{E}}_k(t,s,\xi)\right\|, \left\|\widetilde{\mathcal{E}}_k(s,t,\xi)\right\| \leqslant C_k\left(\frac{1+t}{1+s}\right)^K, \quad t \geqslant s, \tag{3.33}$$

where the constant K is independent of k (and depends only on the first diagonal lower order term F^0). Similarly, we obtain with the same K

$$\left\|\mathcal{E}_k(t,s,\xi)\right\| \leqslant \exp\left(\int_s^t \left\|\operatorname{Im}\bigl(F_{k-1}(\tau,\xi) + R_k(\tau,\xi)\bigr)\right\|\,\mathrm{d}\tau\right) \leqslant C_k'\left(\frac{1+t}{1+s}\right)^K \tag{3.34}$$

for all $t \geqslant s$. Choosing k large, the polynomial decay of the remainder $R_k(t,\xi)$ becomes strong enough to compensate the increasing terms and we obtain the Duhamel representation

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(t,s,\xi)\mathcal{Z}_k(s,\xi) - \mathrm{i}\int_t^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\,\mathrm{d}\theta \tag{3.35}$$

with

$$\mathcal{Z}_k(s,\xi) = \mathrm{I} + \mathrm{i}\int_s^\infty \widetilde{\mathcal{E}}_k(s,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\,\mathrm{d}\theta \lesssim 1. \tag{3.36}$$

Indeed, the integral in (3.35) is convergent for $k > 2K$,

$$\left\| \int_t^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi) R_k(\theta,\xi) \mathcal{E}_k(\theta,s,\xi)\,\mathrm{d}\theta \right\| \lesssim |\xi|^{-k} \int_t^\infty \left(\frac{1+\theta}{1+t}\right)^K \frac{1}{(1+\theta)^{k+1}} \left(\frac{1+\theta}{1+s}\right)^K \mathrm{d}\theta$$

and bounded by $|\xi|^{-k}(1+t)^{K-k}(1+s)^{-K}$. Similarly $\|\mathcal{Z}_k(s,\xi)-\mathrm{I}\| \leqslant |\xi|^{-k}(1+s)^{-k}$ and hence the first term has the lower norm bound $\|\widetilde{\mathcal{E}}_k(t,s,\xi)\| \geqslant (1+t)^K(1+s)^{-K}$ for fixed $|\xi|$. Choosing s big enough depending on $|\xi|$ implies that $\mathcal{E}_k(t,s,\xi)$ is a small perturbation of $\widetilde{\mathcal{E}}_k(t,s,\xi)$.

Assume now that (A3) is violated. Then we find sequences $t_\mu \to \infty$, s_μ, and ξ_μ such that one matrix entry of the integral in (2.12) tends to either ∞ or $-\infty$. We consider the $+\infty$ case, and assume w.l.o.g. that $s_\mu > s$ for sufficiently big s and that the matrix entry corresponds to the first diagonal element. Then with e_1 the first basis vector $\widetilde{\mathcal{E}}_k(t_\mu,s_\mu,\xi_\mu)e_1 \to \infty$ and therefore also $\mathcal{E}(t_\mu,s_\mu,\xi_\mu)e_1 \simeq \mathcal{E}_k(t_\mu,s_\mu,\xi_\mu)N_k(s_\mu,\xi_\mu)M(s_\mu,\xi_\mu)e_1 \to \infty$ which contradicts to the uniform upper bound. Similarly, the $-\infty$ case contradicts to the lower bound and the statement is proven. □

Remark 16. A similar argument does not apply if $\nu > 0$. In this situation the polynomial bound becomes a superpolynomial bound by $\exp(C(\log t)^{1+\nu})$ which cannot be compensated within the diagonalisation hierarchy.

The following results explain why we performed $k > 1$ steps of diagonalisation. In order to prove dispersive estimates later on, we have to control derivatives of $\mathcal{Q}_k(t,x,\xi)$. Formally differentiating (3.30) includes ξ-derivatives of $\widetilde{\mathcal{E}}_k(t,s,\xi)$, which are increasing in t. This can be compensated by better estimates on the remainder $R_k(t,\xi)$. As we will see below, more steps of the diagonalisation hierarchy allow for symbol-like estimates for more derivatives of the entries of $\mathcal{Q}_k(t,x,\xi)$.

Lemma 17. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$, (A2) *and* (A3)*. Then the matrix $\mathcal{Q}_k(t,s,\xi)$ with $2k \leqslant \ell_2$, satisfies for all $|\alpha| \leqslant \min(k-1,\ell_1)$*

$$\left\| \mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,s,\xi) \right\| \lesssim |\xi|^{-|\alpha|} \bigl(\log\bigl(e+|\xi|^{-1}\bigr)\bigr)^{|\alpha|} \tag{3.37}$$

uniformly in $s,t \geqslant \tilde{t}_\xi$. Furthermore, for $|\xi| \leqslant N$ and $|\alpha| \leqslant \min(\frac{k-1}{2},\ell_1)$, we have

$$\left\| \mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,\tilde{t}_\xi,\xi) \right\| \lesssim |\xi|^{-|\alpha|} \bigl(\log\bigl(e+|\xi|^{-1}\bigr)\bigr)^{|\alpha|} \tag{3.38}$$

uniformly in $t \geqslant \tilde{t}_\xi$.

Proof. The statement follows similar to the considerations in [10, Lemma 2.10]. We concentrate on the second estimate, the first one is analogous. Note first, that by differentiating (3.25) we obtain

$$\left\| \mathrm{D}_\xi^\alpha \widetilde{\mathcal{E}}_k(t,\tilde{t}_\xi,\xi) \right\| \lesssim (1+t)^{|\alpha|} \bigl(\log(e+t)\bigr)^{|\alpha|}, \tag{3.39}$$

the logarithmic term arises from differentiating $F_{k-1}(t,\xi)$ in (3.25). This implies that $\mathcal{R}_k(t,t_\xi,\xi)$ satisfies weaker estimates than $R_k(t,\xi)$, for which we have $\mathcal{S}_N^{\ell_1,\ell_2-k-1}\{-k,k+1\}$ estimates. Based on the equivalent to [19, Proposition 11] we obtain

$$\left\| \mathrm{D}_\xi^\alpha \mathcal{R}_k(t,t_\xi,\xi) \right\| \lesssim |\xi|^{-1-|\alpha|}(1+t)^{-2} \bigl(\log(e+t)\bigr)^{|\alpha|} \tag{3.40}$$

for $|\alpha| \leqslant \frac{k-1}{2}$ and $|\alpha| \leqslant \ell_1$. Differentiating the series representation (3.30) term by term, taking into account these estimates and combining them with

$$\left|\mathrm{D}_\xi^\alpha \tilde t_\xi\right| \lesssim \tilde t_\xi |\xi|^{-\alpha} \lesssim |\xi|^{-1-|\alpha|}\big(\log(e+\tilde t_\xi)\big)^{2\nu} \tag{3.41}$$

proves the desired result. □

Remark 18. If we diagonalise perfectly modulo $\mathcal{H}_{N,\nu}\{1\}$ we can construct the corresponding fundamental solution $\mathcal{E}_\infty(t,s,\xi) = \widetilde{\mathcal{E}}_\infty(t,s,\xi)\mathcal{Q}_\infty(t,s,\xi)$ and the previous lemma holds for all multi-indices α. Note, that $\mathcal{H}_{N,\nu}\{1\}$ is invariant under multiplications by $\widetilde{\mathcal{E}}_\infty(t,s,\xi)$.

3.2.5. Estimate in the oscillating sub-zone

It remains to estimate the fundamental solution in the oscillating sub-zone. In this part of the phase space it suffices to apply one step of diagonalisation, i.e. diagonalisation modulo $\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-2}\{-1,2\}$. We construct the fundamental solution $\mathcal{E}_1(t,s,\xi) = \widetilde{\mathcal{E}}_1(t,s,\xi)\mathcal{Q}_1(t,s,\xi)$ of $\mathrm{D}_t - \mathcal{D} - F_1 - R_1$ following the lines of the previous subsection replacing the uniform integrability of the remainder by the estimate

$$\begin{aligned}\int_{t_\xi}^{\tilde t_\xi} \big\|\mathcal{R}_1(t,t_\xi,\xi)\big\|\,\mathrm{d}t &\lesssim \int_{t_\xi}^{\tilde t_\xi} \frac{(\log(e+t))^{2\nu}}{(1+t)^2|\xi|}\,\mathrm{d}t \\ &\sim \frac{(\log(e+t))^{2\nu}}{(1+t)|\xi|}\bigg|_{t_\xi}^{\tilde t_\xi} \leqslant N\big(\log(e+t_\xi)\big)^{\nu}.\end{aligned} \tag{3.42}$$

This implies

$$\big\|\mathcal{Q}_1(t,s,\xi)\big\| \lesssim \exp\big(C'\big(\log(e+t_\xi)\big)^{\nu}\big) \tag{3.43}$$

for $t_\xi \leqslant s \leqslant t \leqslant \tilde t_\xi$. For later use we need estimates for derivatives of $\mathcal{Q}_1$ with respect to ξ.

Lemma 19. *Assume* $(\mathrm{A}1)_{\ell_1,\ell_2}$, $(\mathrm{A}2)$ *and* $(\mathrm{A}3)$, *with* $\ell_2 \geqslant 2$. *Then the matrix* $\mathcal{Q}_1(t,s,\xi)$ *satisfies*

$$\big\|\mathrm{D}_\xi^\alpha \mathcal{Q}_1(t,t_\xi,\xi)\big\| \lesssim |\xi|^{-|\alpha|}\big(\log(e+t)\big)^{(1+2\nu)|\alpha|}\exp\big(C'\big(\log(e+t_\xi)\big)^{\nu}\big) \tag{3.44}$$

for all $|\xi| \leqslant N$ *and all multi-indices* $|\alpha| \leqslant \ell_1$.

Proof. See, e.g., [10, Lemma 2.9]. The proof follows directly by differentiating (3.30) (for $k=1$), using the symbol estimates together with (3.10) and estimates (3.42). □

3.3. Combination of results

We combine the representations of the two previous sections. We distinguish between three cases. If $|\xi| > N$ then only the hyperbolic zone occurs. Hence

$$\mathcal{E}(t,0,\xi) = M^{-1}(t,\xi)N_k^{-1}(t,\xi)\mathcal{E}_k(t,0,\xi)N_k(0,\xi)M(0,\xi). \tag{3.45a}$$

For $|\xi| \leqslant N$ and $t \leqslant t_\xi$ only the pseudo-differential zone is of interest and the fundamental solution is constructed in Section 3.1. For $t \geqslant t_\xi$ we obtain

$$\mathcal{E}(t,0,\xi) = M^{-1}(t,\xi)N_k^{-1}(t,\xi)\mathcal{E}_k(t,t_\xi,\xi)N_k(t_\xi,\xi)M(t_\xi,\xi)\mathcal{E}(t_\xi,0,\xi) \tag{3.45b}$$

with $k=1$ in the oscillating sub-zone and large k and $\tilde{t}_\xi$ in place of t_ξ in the regular sub-zone.

We will bring these representations into a unified form. For this, let us define

$$\varphi_j(t,\xi) = \frac{1}{t}\int_0^t \lambda_j(\tau,\xi)\,\mathrm{d}\tau. \tag{3.46}$$

Then the following statement holds true.

Theorem 20. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$, (A2), (A3) *and* (A4). *Then the fundamental matrix* $\mathcal{E}(t,0,\xi)$ *can be represented as*

$$\mathcal{E}(t,0,\xi) = \sum_{j=1}^m e^{\mathrm{i}t|\xi|\varphi_j(t,\xi/|\xi|)} B_j(t,\xi), \tag{3.47}$$

with matrices $B_j(t,\xi) \in \mathbb{C}^{m\times m}$ *subject to the estimates*

$$\left\| B_j(t,\xi)\right\| \leqslant C\exp\left(C'\left(\log(e+t)\right)^\nu\right) \tag{3.48}$$

in $Z_{pd}(N,\nu)\cup Z_{osc}(N,\nu)$, *and*

$$\left\| \mathrm{D}_\xi^\alpha B_j(t,\xi)\right\| \leqslant C_\alpha|\xi|^{-|\alpha|}\left(\log(e+t)\right)^{(2\nu+1)|\alpha|}\exp\left(C'\left(\log(e+t)\right)^\nu\right) \tag{3.49}$$

for all $|\alpha| \leqslant \min(\ell_1, \ell_2/2-1)$ *in* $Z_{reg}(N,\nu)$.

Proof. Note first, that for $(t,\xi)\in Z_{pd}(N,\nu)\cup Z_{osc}(N,\nu)$, all the terms $e^{\pm\mathrm{i}t|\xi|\phi_j(t,\xi/|\xi|)}$ are uniformly bounded and satisfy the corresponding symbolic estimates. Thus artificially introducing these terms does not destroy our statement.

The estimate in the pseudo-differential zone and the oscillating sub-zone follows directly from Lemma 9 and Eqs. (3.26) and (3.43). Note that for $\nu=0$ we obtain just uniform bounds.

In the regular sub-zone the matrices B_j collect terms from the diagonalisers M, N_k, the matrices $\mathcal{E}(t_\xi,0,\xi)$ and $\mathcal{Q}_k(t,\xi)$ and the terms arising from

$$\exp\left(\mathrm{i}\int_{\max(t_\xi,0)}^t F_k(\tau,\xi)\,\mathrm{d}\tau\right), \tag{3.50}$$

all of which satisfy symbolic estimates due to Lemmas 10, 17 and for the last one due to $F_k \in \mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{0,1\}$. Note further, that (3.41) implies from $N_k(t,\xi)\in\mathcal{S}_{N,\nu}^{\ell_1,\ell_2-k}\{0,0\}$

$$\left\| \mathrm{D}_\xi^\alpha N_k(\tilde{t}_\xi,\xi)\right\| \leqslant C|\xi|^{-|\alpha|}\left(\log(e+\tilde{t}_\xi)\right)^{2\nu|\alpha|} \tag{3.51}$$

which gives the 2ν log-estimate in the regular zone. □

4. Examples

We will give some examples to illustrate the applicability of our previous construction.

4.1. Differential hyperbolic systems

If we consider differential systems, they can always be written as $\mathrm{D}_t U = A(t,\mathrm{D})U$, with

$$A(t,\mathrm{D}) = \sum_{j=1}^n A_j(t)\mathrm{D}_{x_j} + B(t), \tag{4.1}$$

for t-dependent matrices $A_j(t), B(t) \in \mathbb{C}^{m\times m}$. Assumption $(\mathrm{A1})_{\ell_1,\ell_2}$ is satisfied if $A_j(t) \in \mathcal{T}_\nu\{0\}$ and $B(t) \in \mathcal{T}_\nu\{1\}$, where

$$\mathcal{T}_\nu\{\rho\} = \left\{ f \in C^\infty(\mathbb{R}_+)\colon \left|\partial_t^k f(t)\right| \leqslant C_k \left(\frac{1}{1+t}\big(\log(e+t)\big)^\nu\right)^{\rho-k} \right\}, \tag{4.2}$$

together with the assumption that the associated homogeneous matrix

$$\sum_{j=1}^n A_j(t)\xi_j \tag{4.3}$$

has real eigenvalues for all t and ξ. The system is symmetric hyperbolic if the latter matrix is self-adjoint, which means that all matrices $A_j(t)$ are self-adjoint. We denote the eigenvalues of (4.3) by $\lambda_1(t,\xi), \ldots, \lambda_m(t,\xi)$ in ascending order. Assumption (A2) guarantees that they are uniformly distinct if restricted to $\mathbb{R} \times \mathbb{S}^n$.

For the following we assume the system to be symmetric hyperbolic. Then Assumption (A3) simplifies to

$$\sup_{s,t} \left\| \int_s^t \operatorname{diag}\big(M^{-1}(\theta,\xi)\big(\operatorname{Im} B(\theta)\big)M(\theta,\xi)\big)\,\mathrm{d}\theta \right\| < \infty \tag{4.4}$$

for the (0-homogeneous) unitary diagonaliser $M(t,\xi)$ of (4.3), while (A4) reads as

$$\operatorname{Im} B(t) + \gamma(t) I \geqslant 0 \tag{4.5}$$

for a suitable function $\gamma(t)$ satisfying (2.17). Both assumptions are trivially satisfied if we assume $\operatorname{Im} B(t) \in L^1(\mathbb{R}_+)$.

This corresponds to the situation in [3]. Main difference between the results of [3] and our results is, that we do not subsequently require the non-vanishing of the Gaussian curvature of the characteristics. In Theorems 26 and 27 we derive estimates for solutions without this extra assumption. The case of [3] is covered by the case $\gamma = 2$ in Theorem 26.

4.2. Hyperbolic equations of second order

Second order equations under related assumptions are considered in [10,11] and related papers. If we consider the equation

$$\mathrm{D}_t^2 u = \mathrm{D}_t \sum_{j=1}^n \mathrm{D}_{x_j} b_j(t) u + \sum_{1\leqslant i\leqslant j\leqslant n} a_{i,j}(t)\mathrm{D}_{x_i}\mathrm{D}_{x_j} u + c(t)\mathrm{D}_t u + \sum_{j=1}^n d_j(t)\mathrm{D}_{x_j} u + e(t) u \tag{4.6}$$

with $a_{i,j}(t), b_j(t) \in \mathcal{T}_\nu\{0\}$, $c(t), d_j(t) \in \mathcal{T}_\nu\{1\}$ and $e(t) \in \mathcal{T}_\nu\{2\}$, we can rewrite it as 2×2 pseudo-differential system in $U = (h(t,\mathrm{D})u, \mathrm{D}_t u)^T$, $h(t,\xi) \simeq |\xi|_t$ (being ξ-independent within the pseudo-differential zone and smoothly continued to the hyperbolic zone by adding a suitable term from $\mathcal{H}_{N,\nu}\{1\}$),

$$\mathrm{D}_t U = A(t,\mathrm{D})U, \tag{4.7}$$

where within $Z_{hyp}(N,\nu)$ we have

$$\begin{aligned} A(t,\xi) = &\begin{pmatrix} 0 & |\xi| \\ |\xi|^{-1}\sum a_{i,j}(t)\xi_i\xi_j & \sum b_j(t)\xi_j \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ |\xi|^{-1}\sum d_j(t)\xi_j & c(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ |\xi|^{-1}e(t) & 0 \end{pmatrix} \end{aligned} \tag{4.8}$$

modulo lower order terms from $\mathcal{H}_{N,\nu}\{1\}$. The second line gives the terms from $\mathcal{S}_{N,\nu}\{0,1\}$ and $\mathcal{S}_{N,\nu}\{-1,2\}$, respectively. Similarly, we obtain within $Z_{pd}(N,\nu)$

$$\begin{aligned} A(t,\xi) = &\begin{pmatrix} 0 & h(t) \\ h(t)^{-1}\sum a_{i,j}(t)\xi_i\xi_j & \sum b_j(t)\xi_j \end{pmatrix} \\ &+ \begin{pmatrix} \mathrm{D}_t \log h(t) & 0 \\ h(t)^{-1}\sum d_j(t)\xi_j & c(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ h(t)^{-1}e(t) & 0 \end{pmatrix} \end{aligned} \tag{4.9}$$

with $h(t) = h(t,0) = \frac{N}{1+t}(\log(e+t))^\nu$. Note that some of the 'lower order entries' are the dominant terms now.

Assumptions $(\mathrm{A1})_{\ell_1,\ell_2}$ and (A2) are satisfied if we require that eigenvalues of the homogeneous principal part

$$\lambda_\pm(t,\xi) = \frac{1}{2}\sum_{j=1}^n b_j(t)\xi_j \pm \sqrt{\frac{1}{4}\left(\sum_{j=1}^n b_j(t)\xi_j\right)^2 + \sum_{i\leqslant j} a_{i,j}(t)\xi_i\xi_j} \tag{4.10}$$

are real and distinct, i.e., if $b_j(t)$ is real and

$$0 < \left(\sum_{j=1}^n b_j(t)\xi_j\right)^2 + 4\sum_{i\leqslant j} a_{i,j}(t)\xi_i\xi_j, \quad \xi \neq 0. \tag{4.11}$$

Assumption (A3) requires that the integral of

$$\frac{\lambda_\pm(t,\xi)}{\lambda_+(t,\xi)-\lambda_-(t,\xi)}\operatorname{Im} c(t) + \frac{\partial_t\lambda_\pm(t,\xi)}{\lambda_+(t,\xi)-\lambda_-(t,\xi)} + \frac{\operatorname{Im}\sum_i d_i(t)\xi_i}{\lambda_+(t,\xi)-\lambda_-(t,\xi)} \tag{4.12}$$

is uniformly bounded over the hyperbolic zone. This is in particular the case if $\operatorname{Im} c(t)$ and $\operatorname{Im} d_i(t)$ are integrable and the equivalent to assumption (4.16) is satisfied. Furthermore, (A4) becomes a condition on the lower order terms guaranteeing that their influence is dominated by the principle part.

4.3. Hyperbolic equations of higher order

As third example we consider hyperbolic equations of higher order,

$$\mathrm{D}_t^m u + \sum_{j=0}^{m-1} \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(t)\mathrm{D}_t^j \mathrm{D}_x^\alpha u = 0, \tag{4.13}$$

together with the Cauchy data

$$\mathrm{D}_t^j u(0,\cdot) = u_j, \quad j = 0, 1, \dots, m-1. \tag{4.14}$$

Similar to the second order case, this can be brought into pseudo-differential system form.

Then $(\mathrm{A1})_{\ell_1,\ell_2}$ is satisfied if we assume $a_{j,\alpha} \in \mathcal{T}_\nu\{m-j-|\alpha|\}$ and if the roots $\tau = \lambda_j(t,\xi)$ of the homogeneous principal part, i.e., the solutions of

$$\tau^m + \sum_{j=0}^{m-1} \sum_{j+|\alpha|=m} a_{j,\alpha}(t)\tau^j \xi^\alpha = 0, \tag{4.15}$$

are real. For (A2) we require that they are distinct. If we consider equations which are homogeneous of order m, i.e., $a_{j,\alpha}(t) = 0$ for $j+|\alpha| < m$, condition (A3) is equivalent to

$$\max_{1\leqslant j\leqslant m} \sup_{T\geqslant 0} \sup_{\xi\neq 0} \left| \sum_{k\neq j} \int_0^T \frac{\partial_t \lambda_j(t,\xi)}{\lambda_j(t,\xi) - \lambda_k(t,\xi)} \,\mathrm{d}t \right| < \infty, \tag{4.16}$$

while Assumption (A4) is trivially satisfied.

Remark 21. In [6] homogeneous equations of order m were considered under the stronger assumption $\partial_t a_{j,\alpha}(t) \in L^1(\mathbb{R}_+)$ for $j+|\alpha| = m$. This implies, in particular, that (4.16) is satisfied. The representation of solutions is obtained there by the asymptotic integration method, requiring less conditions on the coefficients $a_{j,\alpha}$ (conditions on only the first time derivatives are enough). However, the asymptotic integration method yields a representation of solutions with less control on its amplitudes and an additional loss of regularity in the dispersive estimates, which does not occur with the present method.

5. Dispersive estimates

In this section we are concerned with dispersive estimate for solutions represented in the form of Theorem 20. From now on we assume that $n \geqslant 2$.

First, we give the estimate for low frequencies:

Lemma 22. *Assume* (A4)*. Then solution $U = U(t,x)$ to* (1) *satisfies*

$$\begin{aligned} &\big\| \mathcal{F}^{-1}\big[\big(1-\chi_{reg}(t,\xi)\big)\widehat{U}(t,\xi)\big]\big\|_{L^\infty(\mathbb{R}^n)} \\ &\quad \leqslant C(1+t)^{-n}\big(\log(e+t)\big)^{2n\nu} \exp\big(C'\log(e+t)^\nu\big)\|U_0\|_{L^1(\mathbb{R}^n)} \end{aligned} \tag{5.1}$$

localised to $Z_{pd}(N,\nu) \cup Z_{osc}(N,\nu)$ (for any choice of N).

Proof. Based on the mapping property of the Fourier transform $\mathcal{F}: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ and the Hölder inequality it is sufficient to estimate

$$\left\| \mathcal{E}(t,0,\xi)\chi_{pd+osc}(t,\xi) \right\|_{L^1(\mathbb{R}^n)} \leqslant \left\| \mathcal{E}(t,0,\xi) \right\|_{L^\infty(|\xi|\leqslant\tilde{\xi}_t)} \|\chi_{pd+osc}\|_{L^1(\mathbb{R}^n)}$$

and, therefore, the estimate follows from Lemma 9 and the geometry of the zone. $\square$

For the high-frequency part we at first derive some abstract statements giving L^p–L^q decay rates for oscillatory integrals with a related structure to the ones constructed in Theorem 20,

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(t,\xi))} b(t,\xi) \hat{f}(\xi)\, d\xi,$$

with a real phase function φ and amplitude b. For simplicity, we omit the inclusion of logarithmic terms in assumptions here, later they will give a (for $\nu < 1$ small) change in the decay rates by a simple argument.

We introduce a cut-off function of the form $\psi((1+|t|)\xi)$ for some $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi) \equiv 1$ for $|\xi| \leqslant \frac{1}{2}$, and 0 for $|\xi| \geqslant 1$. We recall that we use the notation $\dot{L}^p_\kappa(\mathbb{R}^n)$ for the homogeneous Sobolev space $\dot{W}^\kappa_p(\mathbb{R}^n)$.

For now we may assume that phase functions satisfy $\varphi(t,\xi) > 0$ uniformly for all $\xi \neq 0$ in the support of $b(t,\xi)$. This can always be achieved by localisation and in combination with adding a linear function to the phase which will not affect estimates. Dispersive estimates for the corresponding Fourier integral operator T_t are determined by the geometric properties of the family of Fresnel surfaces

$$\Sigma^t = \left\{ \xi \in \mathbb{R}^n \setminus \{0\}\colon\ \varphi(t,\xi) = |\xi|\varphi\big(t, \xi/|\xi|\big) = 1 \right\}. \tag{5.2}$$

Following [18] we introduce two indices for such a Fresnel surface Σ, assuming that it is of class C^k with k being sufficiently large. For $u \in \Sigma$ we denote by $\mathcal{T}_u$ the tangent hyperplane to Σ at u. Then for any plane H of dimension 2 which contains u and the normal of Σ at u we denote by $\gamma(\Sigma; u, H)$ the order of contact between the curve $\Sigma \cap H$ and its tangent $H \cap \mathcal{T}_u$. Furthermore, we set

$$\gamma(\Sigma) = \sup_u \sup_H \gamma(\Sigma; u, H), \qquad \gamma_0(\Sigma) = \sup_u \inf_H \gamma(\Sigma; u, H). \tag{5.3}$$

Obviously the definition implies $2 \leqslant \gamma_0(\Sigma) \leqslant \gamma(\Sigma)$. For isotropic problems Σ is a dilation of $\mathbb{S}^{n-1}$ and $\gamma(\mathbb{S}^{n-1}) = \gamma_0(\mathbb{S}^{n-1}) = 2$. Moreover, if the Gaussian curvature of Σ never vanishes we have $\gamma(\Sigma) = 2$.

To control the order of contact of Σ by tangent lines quantitatively, we will introduce another quantity $\varkappa(\Sigma)$. First, assume that Σ is convex and that it is of class $C^{\gamma(\Sigma)+1}$. For $u \in \Sigma$, rotating Σ if necessary, we may assume that it is parameterised by points $\{(y, h(y)),\ y \in \Omega\}$ near u for an open set $\Omega \subset \mathbb{R}^{n-1}$. For $u = (y, h(y))$, let us define

$$\varkappa(\Sigma; u) = \inf_{|\omega|=1} \sum_{j=2}^{\gamma(\Sigma)} \left| \frac{\partial^j}{\partial \rho^j} h(y+\rho\omega)|_{\rho=0} \right|.$$

From the definition of $\gamma(\Sigma)$ it follows that $\varkappa(\Sigma; u) > 0$ for all $u \in \Omega$. Indeed, from the definition of $\gamma(\Sigma; u, H)$ it follows that if ω is such that $y + \rho\omega \in H$, then

$$\varkappa(\Sigma; u, H) = \left| \frac{\partial^{\gamma(\Sigma;u,H)}}{\partial \rho^{\gamma(\Sigma;u,H)}} h(y+\rho\omega)|_{\rho=0} \right| > 0.$$

Now, we clearly have $\sum_{j=2}^{\gamma(\Sigma)} |\frac{\partial^j}{\partial\rho^j} h(y+\rho\omega)|_{\rho=0}| \geqslant \varkappa(\Sigma; u, H)$, and hence we have $\varkappa(\Sigma; u) > 0$ since the set $|\omega| = 1$ is compact. Noticing that $\varkappa(\Sigma; u)$ is a continuous function of u, by compactness of Σ it follows that if we define

$$\varkappa(\Sigma) = \min_{u\in\Sigma} \varkappa(\Sigma; u),$$

then $\varkappa(\Sigma) > 0$.

If Σ is not convex, we define

$$\varkappa_0(\Sigma) = \min_{u\in\Sigma} \sup_{|\omega|=1} \sum_{j=2}^{\gamma_0(\Sigma)} \left| \frac{\partial^j}{\partial\rho^j} h(y+\rho\omega)|_{\rho=0} \right|.$$

Again, we have $\varkappa_0(\Sigma) > 0$.

The quantities $\varkappa(\Sigma^t)$ and $\varkappa_0(\Sigma^t)$ evaluated for all times t will allow us to ensure that if the surfaces degenerate, it will be at least uniformly with respect to t. If the degeneracy of a certain order is not uniform, it may become uniform when we increase integers γ in the formulations below.

For completeness we include the following proposition. It gives a criterion on the convexity of the level sets of the phases, as well as an upper bound on the index γ for systems arising from differential operators.

Proposition 23. *Assume that* $\det A_1(t,\xi)$ *is polynomial in ξ and denote by $\lambda_k(t,\xi)$, $k = 1,\dots,m$, the characteristic roots of the operator* (1)*, ordered by $\lambda_1(t,\xi) < \lambda_2(t,\xi) < \cdots < \lambda_m(t,\xi)$ for $\xi \neq 0$. Suppose that all the Hessians $\nabla_\xi^2 \lambda_k(t,\xi)$ are semi-definite for $\xi \neq 0$. Then there exists a polynomial $\alpha(t,\xi)$ in ξ, of order one, such that $\lambda_{m/2}(t,\xi) < \alpha(t,\xi) < \lambda_{m/2+1}(t,\xi)$ (if m is even) or $\alpha(t,\xi) = \lambda_{(m+1)/2}(t,\xi)$ (if m is odd). Moreover,*[3] *the hypersurfaces $\Sigma_k^t = \{\xi \in \mathbb{R}^n;\ \tilde\varphi_k(t,\xi) = \pm 1\}$ with $\tilde\varphi_k(t,\xi) = \varphi_k(t,\xi) - \alpha(t,\xi)$ $(k \neq (m+1)/2)$ are convex and we have $\gamma(\Sigma_k^t) \leqslant 2\lfloor m/2 \rfloor$.*

By taking the determinant of the system (1) we can reduce the analysis of characteristics to scalar equations. Then this proposition can be readily shown by a modification of the argument in [17], so we omit the details.

We are now ready to estimate the appearing oscillatory integrals. In the sequel, for $r > 0$, by $\lfloor r \rfloor$ we denote its integer part. The proposition below extends Proposition 4.2 in [6] where the limit of $\varphi(t,\xi)$ was assumed to exist as $t \to \infty$, and which deals with amplitudes of type $(0,0)$.

Proposition 24. *Let $\gamma \in \mathbb{N}$. Let T_t be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(t,\xi))} \bigl[1 - \psi\bigl((1+|t|)\xi\bigr)\bigr] b(t,\xi) \hat f(\xi)\,d\xi,$$

where $\varphi(t,\xi)$ is real valued, continuous in t, smooth in $\xi \in \mathbb{R}^n \setminus 0$, homogeneous of order one in ξ, and such that for some $t_0 > 0$ and $C > 0$ we have

$$C^{-1}|\xi| \leqslant \varphi(t,\xi) \leqslant C|\xi| \quad \text{and} \quad \left|\partial_\xi^\alpha \varphi(t,\xi)\right| \leqslant C|\xi|^{1-|\alpha|} \quad \text{for all } t \geqslant t_0,\ \xi \neq 0, \tag{5.4}$$

and all α such that $|\alpha| \leqslant \max\{\gamma+1, \lfloor (n-1)/\gamma \rfloor + 2\}$. Assume that the sets

[3] Here, as in (3.46), we define $\varphi_j(t,\xi) = \frac{1}{t}\int_0^t \lambda_j(\tau,\xi)\,d\tau$.

$$\Sigma^t = \{\xi \in \mathbb{R}^n \setminus 0: \varphi(t,\xi) = 1\} \tag{5.5}$$

are convex for all $t \geqslant t_0$, and assume that $\limsup_{t\to\infty} \gamma(\Sigma^t) \leqslant \gamma$ *and that* $\liminf_{t\to\infty} \varkappa(\Sigma^t) > 0$. *Let us suppose that the amplitude $b(t,\xi)$ satisfies*

$$\left|\partial_\xi^\alpha b(t,\xi)\right| \leqslant C_\alpha |\xi|^{-|\alpha|} \quad \textit{for all } |\alpha| \leqslant \lfloor (n-1)/\gamma \rfloor + 1. \tag{5.6}$$

Let $1 < p \leqslant 2 \leqslant q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $t \geqslant t_0$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \|f\|_{\dot{L}^p_{N_p}(\mathbb{R}^n)},$$

where $N_p = (n - \frac{n-1}{\gamma})(\frac{1}{p} - \frac{1}{q})$.

Proof. Since the L^2-estimate $\|T_t f\|_{L^2} \leqslant C\|f\|_{L^2}$ readily follows from the Plancherel identity, by interpolation we only need to prove that

$$\|\widetilde{T}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^{-\frac{n-1}{\gamma}} \|f\|_{L^1(\mathbb{R}^n)}, \tag{5.7}$$

with $\widetilde{T}_t = T_t \circ |D|^{-N_1}$, where the amplitude $a(t,\xi)$ of $\widetilde{T}_t$ satisfies $|\partial_\xi^\alpha a(t,\xi)| \leqslant C_\alpha |\xi|^{-N_1-|\alpha|}$ for all $|\alpha| \leqslant \lfloor (n-1)/\gamma \rfloor + 1$ and $N_1 = n - \frac{n-1}{\gamma}$. Let us make an additional decomposition

$$\widetilde{T}_t = \widetilde{T_t^{(1)}} + \widetilde{T_t^{(2)}} = \widetilde{T}_t \circ (1 - \psi(D)) + \widetilde{T}_t \circ \psi(D). \tag{5.8}$$

First we will treat the operator $\widetilde{T_t^{(1)}}$. To simplify the notation, for this part we will denote $a(t,\xi)(1 - \psi(\xi))$ by $a(t,\xi)$ again. By using Besov spaces, we can microlocalise the desired estimate (5.7) to spherical layers in the frequency space. Indeed, let $\{\Phi_j\}_{j=0}^\infty$ be the Littlewood–Paley partition of unity, and let

$$\|u\|_{B^s_{p,q}} = \left(\sum_{j=0}^\infty \left(2^{js} \left\|\mathcal{F}^{-1}\Phi_j(\xi)\mathcal{F}u\right\|_{L^p(\mathbb{R}^n)}\right)^q \right)^{1/q}$$

be the norm of the Besov space $B^s_{p,q}$. Then, because of the continuous embeddings $L^p \subset B^0_{p,2}$ for $1 < p \leqslant 2$, and $B^0_{q,2} \subset L^q$ for $2 \leqslant q < +\infty$ (see [1]), it is sufficient to prove the uniform estimate for the operators with amplitudes $a(t,\xi)\Phi_j(\xi)$. Writing $\Phi_j(\xi) = \Phi_j(\xi)\Psi(\frac{\varphi(t,\xi)}{2^j})$ with some function $\Psi \in C_0^\infty(0,\infty)$, we may prove the uniform estimate for operators with amplitudes $a(t,\xi)\Psi(\frac{\varphi(t,\xi)}{2^j})$. Such choice of Ψ is possible due to our assumption (5.4) on $\varphi(t,\xi)$, and we restrict the analysis for large enough t. Let

$$I(t,x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x\cdot\xi + t\varphi(t,\xi))} a(t,\xi) \Psi\left(\frac{\varphi(t,\xi)}{2^j}\right) \mathrm{d}\xi$$

be the kernel of the corresponding operator. Since we easily have the L^2–L^2 estimate by the Plancherel identity, by analytic interpolation we only need to prove the L^1–L^∞ case of (5.7). In turn, this follows from the estimate $|I(t,x)| \leqslant Ct^{-\frac{n-1}{\gamma}}$, with constant C independent of j.

Let $\kappa \in C_0^\infty(\mathbb{R}^n)$ be supported in a ball centred at the origin, with some radius $r > 0$ to be chosen later. We decompose the kernel $I(t,x)$ as

$$\begin{aligned} I(t,x) &= I_1(t,x) + I_2(t,x) \\ &= \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(t,\xi))} a(t,\xi)\kappa\big(t^{-1}x + \nabla_\xi\varphi(t,\xi)\big)\Psi\left(\frac{\varphi(t,\xi)}{2^j}\right) d\xi \\ &\quad + \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(t,\xi))} a(t,\xi)(1-\kappa)\big(t^{-1}x + \nabla_\xi\varphi(t,\xi)\big)\Psi\left(\frac{\varphi(t,\xi)}{2^j}\right) d\xi. \end{aligned} \tag{5.9}$$

We can easily see that $|I_2(t,x)| \leqslant Ct^{-\frac{n-1}{\gamma}}$. In fact, we can show $|I_2(t,x)| \leqslant Ct^{-l}$ for $l = \lfloor (n-1)/\gamma \rfloor + 1$ and then the required estimate simply follows since $l > (n-1)/\gamma$. Indeed, on the support of $1-\kappa$, we have $|x + t\nabla_\xi\varphi(t,\xi)| \geqslant rt > 0$. Thus, integrating by parts with operator $P = \frac{x + t\nabla_\xi\varphi(t,\xi)}{i|x + t\nabla_\xi\varphi(t,\xi)|^2} \cdot \nabla_\xi$, we get

$$I_2(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(t,\xi))} \big(P^*\big)^l \left[a(t,\xi)(1-\kappa)\big(t^{-1}x + \nabla_\xi\varphi(t,\xi)\big)\Psi\left(\frac{\varphi(t,\xi)}{2^j}\right)\right] d\xi. \tag{5.10}$$

Using the fact that $|\partial_\xi^\alpha\varphi(t,\xi)| \leqslant C|\xi|^{1-|\alpha|}$ by (5.4), we readily observe from (5.10) that the required estimate $|I_2(t,x)| \leqslant Ct^{-l}$ holds. Here we also used the condition (5.6) assuring that we can perform the integration by parts $\lfloor (n-1)/\gamma \rfloor + 1$ times. We note that since there is one more $\nabla_\xi\varphi$ involved here, the condition $|\alpha| \leqslant \lfloor (n-1)/\gamma \rfloor + 2$ in (5.4) allows us to integrate by parts l times.

Now we will turn to estimating $I_1(t,x)$. Here we are going to use the structure of the sets Σ^t in (5.5), restricting to t large enough. We recall that (5.4) implies, in particular, that $\varphi(t,\xi) > 0$ for all $\xi \neq 0$. By rotation, we can always microlocalise in some narrow cone around $e_n = (0,\dots,0,1)$, and in this cone we can parameterise

$$\Sigma^t = \big\{\big(y, h_t(y)\big):\ y \in U\big\}$$

for some open $U \subset \mathbb{R}^{n-1}$. In other words, we have $\varphi(t; y, h_t(y)) = 1$, and it follows that h_t is smooth and $\nabla h_t : U \to \nabla h_t(U) \subset \mathbb{R}^{n-1}$ is a homeomorphism. The function h_t is concave if Σ^t is convex. We claim that

$$\big|\partial_y^\alpha h_t(y)\big| \leqslant C_\alpha, \quad \text{for all } y \in U \text{ and large enough } t, \tag{5.11}$$

for all $|\alpha| \leqslant \max\{\gamma + 1, \lfloor (n-1)/\gamma \rfloor + 2\}$. Indeed, let us look at $|\alpha| = 1$ first. From $\varphi(t; y, h_t(y)) = 1$ we get $\nabla_y\varphi + \partial_{\xi_n}\varphi \cdot \nabla h_t(y) = 0$. From (5.4) we have $|\nabla_\xi\varphi| \leqslant C$, so also $|\nabla_y\varphi| \leqslant C$. By Euler's identity we have $\partial_{\xi_n}\varphi(t; e_n) = \varphi(t; e_n) > 0$, so we have $|\partial_{\xi_n}\varphi| \geqslant c > 0$ since we are in a narrow cone around e_n. From this it follows that $|\nabla_y h_t(y)| \leqslant C$ for all $y \in U$ and t large enough. A similar argument proves the boundedness of higher order derivatives in (5.11).

Now, let us turn to analyse the structure of the sets Σ^t. We have the Gauss map

$$G : \Sigma^t \ni \zeta \mapsto \frac{\nabla_\zeta\varphi(t;\zeta)}{|\nabla_\zeta\varphi(t;\zeta)|} \in \mathbb{S}^{n-1},$$

and for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ near the point $-\nabla_\zeta\varphi(t; e_n)$ we define $z_t \in U$ by $(z_t, h_t(z_t)) = G^{-1}(-x/|x|)$. Then $(-\nabla_y h_t(y), 1)$ is normal to Σ^t at $(y, h_t(y))$, so we get

$$-\frac{x}{|x|} = \frac{(-\nabla_y h_t(z_t), 1)}{|(-\nabla_y h_t(z_t), 1)|} \quad \text{and} \quad \frac{x'}{x_n} = -\nabla_y h_t(z_t).$$

Making change of variables $\xi = (\tilde{\lambda} y, \tilde{\lambda} h_t(y))$ and using $\varphi(t,\xi) = \tilde{\lambda}$, we get

$$\begin{aligned}
I_1(t,x) &= \int_0^\infty \int_U \mathrm{e}^{\mathrm{i}\tilde{\lambda}(x'\cdot y + x_n h_t(y) + t)} a\big(t, \tilde{\lambda} y, \tilde{\lambda} h_t(y)\big) \Psi\left(\frac{\tilde{\lambda}}{2^j}\right) \kappa_0(t,x,y) \left|\frac{\mathrm{d}\xi}{\mathrm{d}(\tilde{\lambda}, y)}\right| \mathrm{d}y\, \mathrm{d}\tilde{\lambda} \\
&= \int_0^\infty \int_U \mathrm{e}^{\mathrm{i}\tilde{\lambda}(-x_n \nabla_y h_t(z_t)\cdot y + x_n h_t(y) + t)} \big[\tilde{\lambda}^l a\big(t, \tilde{\lambda} y, \tilde{\lambda} h_t(y)\big)\big] \tilde{\lambda}^{n-1-l} \\
&\quad \times \Psi\left(\frac{\tilde{\lambda}}{2^j}\right) \kappa_0(t,x,y) \chi(t,y)\, \mathrm{d}y\, \mathrm{d}\tilde{\lambda} \\
&= \int_0^\infty \int_U \mathrm{e}^{\mathrm{i}\lambda(-\nabla_y h_t(z_t)\cdot y + h_t(y) + t x_n^{-1})} \tilde{a}\big(t, x_n, \lambda y, \lambda h_t(y)\big) \lambda^{n-1-l} \\
&\quad \times \Psi\left(\frac{\lambda}{2^j x_n}\right) x_n^{-n+1+l-1} \kappa_0(t,x,y) \chi(t,y)\, \mathrm{d}y\, \mathrm{d}\lambda,
\end{aligned}$$

where $\kappa_0(t,x,y) = \kappa(t^{-1}x + \nabla_\xi \varphi(t; y, h_t(y)))$, and

$$\tilde{a}\big(t, x_n, \lambda y, \lambda h_t(y)\big) = \big(x_n^{-1}\lambda\big)^l a\big(t, x_n^{-1}\lambda y, x_n^{-1}\lambda h_t(y)\big),$$

and where we made a change $\tilde{\lambda} = x_n^{-1}\lambda$ in the last equality. Here also we used $|\frac{\mathrm{d}\xi}{\mathrm{d}(\tilde{\lambda}, y)}| = \tilde{\lambda}^{n-1}\chi(t,y)$, where $\chi(t,y)$ and its derivatives with respect to y are bounded because of (5.11).

If we choose r in the definition of the cut-off function κ sufficiently small, then on its support we have $|x| \approx |x_n| \approx t$, and we can estimate

$$\begin{aligned}
\big|I_1(t,x)\big| &\leqslant C t^{-n+l} \int_0^\infty \left| J(\lambda, z_t) \Psi\left(\frac{\lambda}{2^j t}\right) \lambda^{n-1-l} \right| \mathrm{d}\lambda \\
&= C t^{-n+l} 2^{j(n-l)} \int_0^\infty \left| J\big(2^j \lambda, z_t\big) \Psi\left(\frac{\lambda}{t}\right) \lambda^{n-1-l} \right| \mathrm{d}\lambda, \qquad (5.12)
\end{aligned}$$

with

$$J(\lambda, z_t) = \int_U \mathrm{e}^{\mathrm{i}\lambda(-\nabla_y h_t(z_t)\cdot y + h_t(y) + t x_n^{-1})} \tilde{a}\big(t, x_n, \lambda y, \lambda h_t(y)\big) \kappa_0(t,x,y) \chi(t,y)\, \mathrm{d}y.$$

We will show that

$$\big|J(\lambda, z_t)\big| \leqslant C(1+\lambda)^{-\frac{n-1}{\gamma}}, \quad \lambda > 0. \qquad (5.13)$$

Then, if we take $l = n - \frac{n-1}{\gamma}$, and use (5.12) and (5.13), we get

$$
\begin{aligned}
\left|I_1(t,x)\right| &\leqslant Ct^{-\frac{n-1}{\gamma}} 2^{j\frac{n-1}{\gamma}} \int_0^\infty \left(2^j\lambda\right)^{-\frac{n-1}{\gamma}} \Psi\left(\frac{\lambda}{t}\right) \lambda^{\frac{n-1}{\gamma}-1}\,\mathrm{d}\lambda \\
&= Ct^{-\frac{n-1}{\gamma}} \int_0^\infty \lambda^{-1}\Psi\left(\frac{\lambda}{t}\right)\mathrm{d}\lambda = Ct^{-\frac{n-1}{\gamma}} \int_0^\infty \lambda^{-1}\Psi(\lambda)\,\mathrm{d}\lambda \\
&\leqslant Ct^{-\frac{n-1}{\gamma}},
\end{aligned}
\tag{5.14}
$$

which is the desired estimate for $I_1(t,x)$.

Let us now prove (5.13). We note first that with this choice of l we have

$$
\left|\partial_y^\alpha \tilde{a}\right| \leqslant C \quad \text{for all } |\alpha| \leqslant \lfloor (n-1)/\gamma \rfloor + 1. \tag{5.15}
$$

Now, estimate (5.13) follows from Theorem 29 in Appendix A. Indeed, we can write $J(\lambda, z_t)$ in polar coordinates (ρ,ω) with $y = \rho\omega + z_t$, so that

$$
J(\lambda, z_t) = e^{\mathrm{i}\lambda(tx_n^{-1}+h_t(z_t))} \int_{\mathbb{S}^{n-2}} \int_0^\infty e^{\mathrm{i}\lambda F(\rho,z_t,\omega)} \beta(\rho, z_t, \omega)\rho^{n-2}\,\mathrm{d}\rho\,\mathrm{d}\omega, \tag{5.16}
$$

with

$$
\begin{aligned}
F(\rho, z_t, \omega) &= h_t(\rho\omega + z_t) - h_t(z_t) - \rho\nabla_y h_t(z_t)\cdot\omega, \\
\beta(\rho, z_t, \omega) &= \tilde{a}\big(t, x_n, \lambda(\rho\omega + z_t), \lambda h_t(\rho\omega + z_t)\big)\kappa_0(t, x, \rho\omega + z_t)\chi(t, \rho\omega + z_t),
\end{aligned}
\tag{5.17}
$$

where we can assume in addition that $\chi = 0$ unless $\rho\omega + z_t \in U$, so both ρ and ω vary over bounded sets. Now, we want to apply Theorem 29 to obtain estimate (5.13). The function F in (5.17) satisfies condition (F3) of Theorem 29 because of the definition of the convex index γ and because h_t is concave. From the assumption $\liminf_{t\to\infty} \varkappa(\Sigma_t) > 0$ it follows that function F satisfies property (F2) of Theorem 29. One can readily see that the other conditions of Theorem 29 are satisfied, implying (5.13).

We now prove the estimates for the integral $\widetilde{T_t^{(2)}}$ in the decomposition (5.8). This proof is essentially similar to that for $\widetilde{T_t^{(1)}}$ with a few differences that we will point out here. Again, by interpolation, it is sufficient to prove estimate

$$
\left\|\widetilde{T_t^{(2)}} f\right\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^{-\frac{n-1}{\gamma}} \|f\|_{L^1(\mathbb{R}^n)}, \tag{5.18}
$$

with amplitude $a(t,\xi)$ satisfying

$$
\left|\partial_\xi^\alpha a(t,\xi)\right| \leqslant C_\alpha |\xi|^{-N_1-|\alpha|} \quad \text{for all } |\alpha| \leqslant \lfloor (n-1)/\gamma \rfloor + 1, \tag{5.19}
$$

with $N_1 = n - \frac{n-1}{\gamma}$.

We continue to estimate the integrals $I_1(t,x)$ and $I_2(t,x)$ in the decomposition (5.9) corresponding to the integral $\widetilde{T_t^{(2)}}$. In general, since for $\widetilde{T_t^{(2)}}$ we work with low frequencies $|\xi| < 1$ only, no Besov space decomposition is necessary, so we do not need to introduce functions Ψ and Φ_j, and we can take $\Psi = 1$.

The additional complications are related to the fact that in principle derivatives of the amplitude of the operator $\widetilde{T_t^{(2)}}$ may introduce an additional growth with respect to t. In the estimate for $I_2(t,x)$ we performed integration by parts with operator P. Now after the integration by parts the amplitude of this integral in (5.10) is

$$\left(P^*\right)^l\left[\left[1-\psi\left((1+|t|)\xi\right)\right]a(t,\xi)(1-\kappa)\left(t^{-1}x+\nabla_\xi\varphi(t,\xi)\right)\Psi\left(\frac{\varphi(t,\xi)}{2^j}\right)\right].$$

Then, if any of the ξ-derivatives falls on $[1-\psi((1+|t|)\xi)]$, we get an extra factor t which is cancelled with t^{-1} in the definition of P. However, in this case we can then restrict to the support of $\nabla\psi$ which is contained in the ball with radius $(1+|t|)^{-1}$, so we are in the situation of very low frequencies $|\xi|\leqslant Ct^{-1}$ again. Consequently, for this integral we actually get a better decay rate of Lemma 22. If none of the derivatives in $(P^*)^l$ fall on $[1-\psi((1+|t|)\xi)]$, the argument is the same as in the proof of the estimate for $I_2(t,x)$ for the integral $\widetilde{T_t^{(1)}}$.

Let us now analyse the term $I_1(t,x)$ corresponding to $\widetilde{T_t^{(2)}}$. Recall now that in the process of estimating $I_1(t,x)$ corresponding to $\widetilde{T_t^{(1)}}$, we made a change of variables $\tilde\lambda=x_n^{-1}\lambda$. As it was then pointed out, if r in the definition of the cut-off function κ is chosen sufficiently small, on its support we have $|x_n|\approx|t|$. On the other hand, we have $|\xi|\approx\tilde\lambda$ by the definition of $\tilde\lambda$, since we may assume that $\varphi(t,\xi)$ is strictly positive for $\xi\neq 0$. It then follows that $(1+|t|)\xi\approx\tilde\lambda|x_n|\approx\lambda$, and so the change of variables $\tilde\lambda=x_n^{-1}\lambda$ changes $[1-\psi((1+|t|)\xi)]$ into $[1-\tilde\psi(\lambda)]$ in the amplitude of $I_1(t,x)$. Justifying this argument, we can then continue as in the case of $\widetilde{T_t^{(1)}}$. The crucial condition for the use of Theorem 29 is the boundedness of derivatives of $\tilde a$ in (5.15). Here, every differentiation of a with respect to y introduces a factor $x_n^{-1}\lambda$ which is then cancelled in view of assumption (5.19). It follows that $\lfloor(n-1)/\gamma\rfloor+1$ y-derivatives of $\tilde a$ are bounded, implying the conclusion of Theorem 29. This yields estimate (5.18) in the way that is similar to the proof of the same estimate for $\widetilde{T_t^{(1)}}$. $\square$

We now give an analogue of Proposition 24 without the convexity assumption. Naturally, in this case we get slower decay rates at infinity.

Proposition 25. *Let $\gamma_0\in\mathbb{N}$. Let T_t be an operator defined by*

$$T_tf(x)=\int\limits_{\mathbb{R}^n}e^{\mathrm{i}(x\cdot\xi+t\varphi(t,\xi))}\left[1-\psi\left((1+|t|)\xi\right)\right]b(t,\xi)\hat f(\xi)\,\mathrm{d}\xi,$$

where $\varphi(t,\xi)$ is real valued, continuous in t, smooth in $\xi\in\mathbb{R}^n\setminus 0$, homogeneous of order one in ξ, and such that for some $t_0>0$ and $C>0$ we have

$$C^{-1}|\xi|\leqslant\varphi(t,\xi)\leqslant C|\xi|\quad\textit{and}\quad\left|\partial_\xi^\alpha\varphi(t,\xi)\right|\leqslant C|\xi|^{1-|\alpha|}\quad\textit{for all } t\geqslant t_0,\ \xi\neq 0,\tag{5.20}$$

and all α such that $|\alpha|\leqslant\gamma_0+1$. Assume that the sets

$$\Sigma^t=\left\{\xi\in\mathbb{R}^n\setminus 0\colon\ \varphi(t,\xi)=1\right\}\tag{5.21}$$

satisfy $\limsup_{t\to\infty}\gamma_0(\Sigma^t)\leqslant\gamma_0$ and $\liminf_{t\to\infty}\varkappa_0(\Sigma^t)>0$. Let us suppose that the amplitude $b(t,\xi)$ satisfies

$$\left|\partial_\xi^\alpha b(t,\xi)\right|\leqslant C_\alpha|\xi|^{-|\alpha|}\quad\textit{for all } |\alpha|\leqslant 1.\tag{5.22}$$

Let $1<p\leqslant 2\leqslant q<\infty$ be such that $\frac1p+\frac1q=1$. Then for all $t\geqslant t_0$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}\|f\|_{\dot{L}^p_{N_p}(\mathbb{R}^n)},$$

where $N_p = (n - \frac{1}{\gamma_0})(\frac{1}{p} - \frac{1}{q})$.

Proof. Let us show how the proof of Proposition 25 differs from the proof of Proposition 24. First, we need to prove that $|I(t,x)| \leqslant Ct^{-\frac{1}{\gamma_0}}$, $t \geqslant t_0$, for $I_1(t,x)$ as in (5.9). We note that $\gamma_0 + 1 \geqslant 2$ (actually, as we observed before, we must have $\gamma_0 \geqslant 2$), so to prove the estimate for $I_2(t,x)$ we can show that $|I_2(t,x)| \leqslant Ct^{-1}$. This can be done by integrating by parts with the same operator P one time, and using (5.22) instead of (5.6). As for the proof of the estimate for $I_1(t,x)$, we can reason in the same way as in Proposition 24 to arrive at the estimate (5.12), i.e.,

$$\left|I_1(t,x)\right| \leqslant Ct^{-n+l}2^{j(n-l)} \int_0^\infty \left| J\left(2^j\lambda, z_t\right)\Psi\left(\frac{\lambda}{t}\right)\lambda^{n-1-l}\right| \mathrm{d}\lambda$$

with the same operator

$$J(\lambda, z_t) = \int_U e^{\mathrm{i}\lambda(-\nabla_y h_t(z_t)\cdot y + h_t(y) + tx_n^{-1})}\tilde{a}\big(t, x_n, \lambda y, \lambda h_t(y)\big)\kappa_0(t,x,y)\chi(t,y)\,\mathrm{d}y.$$

Now, instead of (5.13) we will show that

$$\left|J(\lambda, z_t)\right| \leqslant C(1+\lambda)^{-\frac{1}{\gamma_0}}, \quad \lambda > 0. \tag{5.23}$$

Then, taking $l = n - \frac{1}{\gamma_0}$, we get the estimate $|I_1(t,x)| \leqslant Ct^{-\frac{1}{\gamma_0}}$ in the same way as in estimate (5.14). Now, estimate (5.23) follows from Theorem 29 in Appendix A with $N = 1$. Indeed, let us write $J(\lambda, z_t)$ in the form (5.16)–(5.17) with phase

$$F(\rho, z_t, \omega) = h_t(\rho\omega + z_t) - h_t(z_t) - \rho\nabla_z h_t(z_t)\cdot\omega.$$

Now, by rotation we may assume that in some direction, say $e_1 = (1, 0, \ldots, 0)$, we have by definition of the index γ_0 that

$$\gamma_0 = \min\left\{k \in \mathbb{N}\colon\ \partial^k_{\omega_1} F(\rho, z_t, \omega)|_{\omega_1=0} \neq 0\right\}.$$

Then by taking $N = 1$ and $y = \omega_1$ in Theorem 29, we get the required estimate (5.23). Conditions $\limsup_{t\to\infty}\gamma_0(\Sigma^t) \leqslant \gamma_0$ and $\liminf_{t\to\infty}\varkappa_0(\Sigma^t) > 0$ ensure that the dependence on t in this argument is uniform, so that the constants are also uniform in t. Finally, the estimate for $I_2(t,x)$ differs from that for $I_1(t,x)$ in exactly the same way as in the proof of Proposition 24, so we can omit the repetition of the argument there. □

Suitable L^p–L^q estimates for solutions to (1) under Assumptions (A1) to (A4) can now be expressed as corollary of these two propositions (depending whether all Fresnel surfaces are convex or not).

Theorem 26. *Assume* $(\mathrm{A1})_{\ell_1,\ell_2}$, (A2), (A3) *and* (A4). *Assume further that the Fresnel surfaces* Σ^t_k, $k = 1, \ldots, m$, *are convex for* $t \geqslant t_0$ *for some time* t_0 *and that* $\limsup_{t\to\infty}\gamma(\Sigma^t_k) \leqslant \gamma$ *with* $\liminf_{t\to\infty}\varkappa(\Sigma^t_k) > 0$, *together with* $\lfloor (n-1)/\gamma \rfloor \leqslant \min(\ell_1 - 1, \ell_2/2 - 2)$. *Then solutions to* (1) *satisfy the* L^p–L^q *estimate*

$$\big\|U(t,\cdot)\big\|_{L^q(\mathbb{R}^n)} \leqslant C_{pq\epsilon}(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})+\epsilon}\|U_0\|_{W^{p,r_p}(\mathbb{R}^n)},$$

for $p \in (1,2]$, $pq = p+q$, $r_p = n(\frac{1}{p}-\frac{1}{q})$, and with $\epsilon > 0$ arbitrary small if $\nu \in [0,1)$ and for some $\epsilon > 0$ if $\nu = 1$.

Proof. Due to Theorem 20 solutions to (1) microlocalised to $Z_{reg}(N,\nu)$ have Fourier integral representations of the kind used in Proposition 24 after multiplication with $t^{-\epsilon}$ (this multiplication makes symbol estimates uniform in t as it eliminates the occurring sub-polynomial/polynomial terms in the estimates from Theorem 20). It remains to check the assumptions of Proposition 24. The phase functions $\phi_j(t,\xi)$ are averages over $\lambda_j(t,\xi)$ and therefore homogeneous and bounded from below and from above, in view of Propositions 3 and 23. Similar for the bounds of their derivatives in ξ. The amplitudes satisfy the assumptions provided ℓ_1 and ℓ_2 are big enough, i.e., $\lfloor (n-1)/\gamma \rfloor + 1 \leqslant \min(\ell_1, \ell_2/2-1)$, combining (3.49) and (5.6).

Finally, combining the microlocal estimate with Lemma 22 and Sobolev embedding theorem to estimate small times concludes the proof. □

We will give one example. If we follow Section 4.3 and consider homogeneous hyperbolic equations of higher order (4.13),

$$\mathrm{D}_t^m u + \sum_{j+|\alpha|=m} a_{j,\alpha}(t)\mathrm{D}_t^j \mathrm{D}_x^\alpha u = 0, \qquad \mathrm{D}_t^j u(0,\cdot) = u_j, \quad j = 0,1,\ldots,m-1, \tag{5.24}$$

for $a_{j,\alpha} \in \mathcal{T}_\nu\{0\}$, satisfying the assumption of uniform strict hyperbolicity in combination with (4.16), then Theorem 26 applies and dispersive type estimates depend on geometric properties of the Fresnel surfaces associated to the problem. Hence, if the problem is rotationally invariant all Fresnel surfaces are given by spheres and we obtain $\gamma(\Sigma_k^t) = 2$. The uniformity condition $\liminf_{t\to\infty} \varkappa(\Sigma_k^t) > 0$ is satisfied if the spheres stay bounded which is the case if $\varphi(t,\xi)$ is uniformly bounded away from zero. In this case the L^p–L^q estimate

$$\sum_{j=0}^{m-1} \big\| |\mathrm{D}|^{m-j-1}\mathrm{D}_t^j u(t,\cdot)\big\|_{L^q} \leqslant C_{p,\epsilon}(1+t)^{\epsilon-\frac{n-1}{2}} \sum_{j=0}^{m-1} \big\| |\mathrm{D}|^{m-j-1} u_j \big\|_{W^{p,r_p}} \tag{5.25}$$

for $1 < p \leqslant 2$, $pq = p+q$, $r_p = n(\frac{1}{p}-\frac{1}{q})$ and with arbitrarily small $\epsilon > 0$ follows. If we drop rotational invariance, examples for $\gamma(\Sigma_k^t) \in \{2,3,\ldots,2\lfloor m/2 \rfloor\}$ can be constructed in analogy and give corresponding weaker decay rates.

If some of the surfaces fail to be convex, decay rates can be much weaker.

Theorem 27. *Assume* (A1)$_{1,2}$, (A2), (A3) *and* (A4), *and assume also that* $\limsup_{t\to\infty} \gamma_0(\Sigma_k^t) \leqslant \gamma_0$ *with* $\liminf_{t\to\infty} \varkappa_0(\Sigma_k^t) > 0$. *Then solutions to* (1) *satisfy the L^p–L^q estimate*

$$\big\|U(t,\cdot)\big\|_{L^q(\mathbb{R}^n)} \leqslant C_{pq\epsilon}(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})+\epsilon}\|U_0\|_{W^{p,r_p}(\mathbb{R}^n)}$$

for $p \in (1,2]$, $pq = p+q$, $r_p = n(\frac{1}{p}-\frac{1}{q})$, and with $\epsilon > 0$ arbitrary small if $\nu \in [0,1)$ and for some $\epsilon > 0$ if $\nu = 1$.

Remark 28. In comparison to results on scalar second order equations due to Reissig and co-authors, e.g., [7,9] or [10], we observe an ϵ-loss of decay even for the case $\nu = 0$. This is due to the ξ-dependence of the term F_0 in general. If $\nu = 0$ the choice $\epsilon = 0$ can be made in both Theorems 26 and 27 provided that $\nabla_\xi F_0(t,\xi) = 0$.

Appendix A. The multi-dimensional van der Corput lemma

We now give the multi-dimensional version of the van der Corput lemma used in the essential way in the proof of Proposition 24, as well as in the proof of Proposition 25.

Theorem 29. *(See [12].) Consider the oscillatory integral*

$$I(\lambda,\nu)=\int_{\mathbb{R}^N} e^{i\lambda\Phi(x,\nu)}a(x,\nu)\chi(x)\,dx,$$

where $N\geqslant 1$ and ν is a parameter. Let $\gamma\geqslant 2$ be an integer. Assume that

(A1) *there exists a sufficiently small $\delta>0$ such that $\chi\in C_0^\infty(B_{\delta/2}(0))$, where $B_{\delta/2}(0)$ is the ball with radius $\delta/2$ around 0;*
(A2) *$\Phi(x,\nu)$ is a complex-valued function such that $\operatorname{Im}\Phi(x,\nu)\geqslant 0$ for all $x\in\operatorname{supp}\chi$ and all parameters ν;*
(A3) *for some fixed $z\in\operatorname{supp}\chi$, the function*

$$F(\rho,\omega,\nu):=\Phi(z+\rho\omega,\nu),\quad |\omega|=1,$$

satisfies the following conditions. Assume that for each $\mu=(\omega,\nu)$, function $F(\cdot,\mu)$ is of class $C^{\gamma+1}$ on $\operatorname{supp}\chi$, and let us write its γ-th order Taylor expansion in ρ at 0 as

$$F(\rho,\mu)=\sum_{j=0}^{\gamma}a_j(\mu)\rho^j+R_{\gamma+1}(\rho,\mu),$$

where $R_{\gamma+1}$ is the remainder term. Assume that we have
(F1) *$a_0(\mu)=a_1(\mu)=0$ for all μ;*
(F2) *there exists a constant $C>0$ such that $\sum_{j=2}^{\gamma}|a_j(\mu)|\geqslant C$ for all μ;*
(F3) *for each μ, $|\partial_\rho F(\rho,\mu)|$ is increasing in ρ for $0<\rho<\delta$;*
(F4) *for each $k\leqslant\gamma+1$, $\partial_\rho^k F(\rho,\mu)$ is bounded uniformly in $0\leqslant\rho<\delta$ and μ;*

(A4) *for each multi-index α of length $|\alpha|\leqslant[\frac{N}{\gamma}]+1$, there exists a constant $C_\alpha>0$ such that $|\partial_x^\alpha a(x,\nu)|\leqslant C_\alpha$ for all $x\in\operatorname{supp}\chi$ and all parameters ν.*

Then there exists a constant $C=C_{N,\gamma}>0$ such that

$$\bigl|I(\lambda,\nu)\bigr|\leqslant C(1+\lambda)^{-\frac{N}{\gamma}}\quad \text{for all } \lambda\in[0,\infty) \text{ and all parameters } \nu.$$

References

[1] J. Bergh, J. Löfström, Interpolation Spaces, Springer, 1976.
[2] P. Brenner, On L^p–L^q estimates for the wave equation, Math. Z. 145 (1975) 251–254.
[3] M. d'Abbico, S. Lucente, G. Taglialatela, L^p–L^q estimates for regularly linear hyperbolic systems, Adv. Differential Equations 14 (2009) 801–834; Adv. Differential Equations 16 (2011) 199–200, Erratum.
[4] F. Hirosawa, J. Wirth, C^m-theory of damped wave equations with stabilisation, J. Math. Anal. Appl. 343 (2008) 1022–1035.
[5] T. Matsuyama, M. Ruzhansky, Time decay for hyperbolic equations with homogeneous symbols, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 915–919.
[6] T. Matsuyama, M. Ruzhansky, Asymptotic integration and dispersion for hyperbolic equations, Adv. Differential Equations 15 (2010) 721–756.
[7] M. Reissig, K. Yagdjian, L_p–L_q estimates for the solutions of hyperbolic equations of second order with time-dependent coefficients — oscillations via growth, preprint 98-5, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg, 1998.

[8] M. Reissig, K. Yagdjian, The L^p–L^q decay estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients, Math. Nachr. 214 (2000) 71–104.
[9] M. Reissig, K. Yagdjian, About the influence of oscillations on Strichartz type decay estimates, Rend. Semin. Mat. Univ. Politec. Torino 58 (2000) 375–388.
[10] M. Reissig, J. Smith, L^p–L^q estimate for wave equations with bounded time dependent coefficient, Hokkaido Math. J. 34 (2005) 541–586.
[11] M. Reissig, Optimality for the asymptotic behaviour of the energy for wave models, in: M. Ruzhansky, J. Wirth (Eds.), Modern Aspects of Partial Differential Equations, Birkhäuser, 2011.
[12] M. Ruzhansky, Pointwise van der Corput lemma for functions of several variables, Funct. Anal. Appl. 43 (2009) 75–77.
[13] M. Ruzhansky, J. Smith, Global time estimates for solutions to equations of dissipative type, in: Journées "Équations aux Dérivées Partielles", Exp. No. XII, Ecole Polytech., Palaiseau, 2005.
[14] M. Ruzhansky, J. Smith, Dispersive and Strichartz Estimates for Hyperbolic Equations with Constant Coefficients, MSJ Mem., vol. 22, Mathematical Society of Japan, Tokyo, 2010.
[15] M. Ruzhansky, J. Wirth, Dispersive estimates for T-dependent hyperbolic systems, Rend. Semin. Mat. Univ. Politec. Torino 66 (2008) 339–349.
[16] R.S. Strichartz, A priori estimates for the wave equation and some applications, J. Funct. Anal. 5 (1970) 218–235.
[17] M. Sugimoto, A priori estimates for higher order hyperbolic equations, Math. Z. 215 (1994) 519–531.
[18] M. Sugimoto, Estimates for hyperbolic equations with non-convex characteristics, Math. Z. 222 (1996) 521–531.
[19] J. Wirth, Wave equations with time-dependent dissipation I. Non-effective dissipation, J. Differential Equations 222 (2006) 487–514.
[20] J. Wirth, Diagonalisation schemes and applications, Ann. Mat. Pura Appl. 189 (2010) 571–590.
[21] J. Wirth, Energy inequalities and dispersive estimates for wave equations with time-dependent coefficients, Rend. Mat. Univ. Trieste (Suppl.) 42 (2010) 205–219.
[22] K. Yagdjian, The Cauchy Problem for Hyperbolic Operators: Multiple Characteristics, Micro-Local Approach, Math. Topics, vol. 12, Akademie Verlag, Berlin, 1997.